SYMMETRIC CRYSTALS FOR $\mathfrak{gl}_\infty$

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ABSTRACT. In the preceding paper, we formulated a conjecture on the relations between certain classes of irreducible representations of affine Hecke algebras of type B and symmetric crystals for $\mathfrak{gl}_\infty$. In the present paper, we prove the existence of the symmetric crystal and the global basis for $\mathfrak{gl}_\infty$.

CONTENTS

1. INTRODUCTION

Lascoux-Leclerc-Thibon ([LLT]) conjectured the relations between the representations of Hecke algebras of *type A* and the crystal bases of the affine Lie algebras of type A. Then, S. Ariki ([A]) observed that it should be understood in the setting of affine Hecke algebras and proved the LLT conjecture in a more general framework. Recently, we presented the

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notion of symmetric crystals and conjectured that certain classes of irreducible representations of the affine Hecke algebras of *type B* are described by symmetric crystals for $\mathfrak{gl}_\infty$ ([EK]).

The purpose of the present paper is to prove the existence of symmetric crystals in the case of $\mathfrak{gl}_\infty$.

Let us recall the Lascoux-Leclerc-Thibon-Ariki theory. Let $\mathrm{H}_n^{\mathrm{A}}$ be the affine Hecke algebra of type A of degree n. Let $\mathrm{K}_n^{\mathrm{A}}$ be the Grothendieck group of the abelian category of finite-dimensional $\mathrm{H}_n^{\mathrm{A}}$-modules, and $\mathrm{K}^{\mathrm{A}} = \oplus_{n\geqslant 0}\mathrm{K}_n^{\mathrm{A}}$. Then it has a structure of Hopf algebra by the restriction and the induction. The set $I = \mathbb{C}^*$ may be regarded as a Dynkin diagram with I as the set of vertices and with edges between $a \in I$ and ap_1^2. Here p_1 is the parameter of the affine Hecke algebra usually denoted by q. Let $\mathfrak{g}_I$ be the associated Lie algebra, and $\mathfrak{g}_I^-$ the unipotent Lie subalgebra. Let U_I be the group associated to $\mathfrak{g}_I^-$. Hence $\mathfrak{g}_I$ is isomorphic to a direct sum of copies of $A_\ell^{(1)}$ if p_1^2 is a primitive ℓ-th root of unity and to a direct sum of copies of $\mathfrak{gl}_\infty$ if p_1 has an infinite order. Then $\mathbb{C}\otimes\mathrm{K}^{\mathrm{A}}$ is isomorphic to the algebra $\mathscr{O}(U_I)$ of regular functions on U_I. Let $U_q(\mathfrak{g}_I)$ be the associated quantized enveloping algebra. Then $U_q^-(\mathfrak{g}_I)$ has an upper global basis $\{\mathrm{G}^{\mathrm{up}}(b)\}_{b\in\mathrm{B}(\infty)}$. By specializing $\bigoplus \mathbb{C}[q,q^{-1}]\mathrm{G}^{\mathrm{up}}(b)$ at $q=1$, we obtain $\mathscr{O}(U_I)$. Then the LLTA-theory says that the elements associated to irreducible H^{A}-modules corresponds to the image of the upper global basis.

In [EK], we gave analogous conjectures for affine Hecke algebras of type B. In the type B case, we have to replace $U_q^-(\mathfrak{g}_I)$ and its upper global basis with symmetric crystals (see § 2.3). It is roughly stated as follows. Let $\mathrm{H}_n^{\mathrm{B}}$ be the affine Hecke algebra of type B of degree n. Let $\mathrm{K}_n^{\mathrm{B}}$ be the Grothendieck group of the abelian category of finite-dimensional modules over $\mathrm{H}_n^{\mathrm{B}}$, and $\mathrm{K}^{\mathrm{B}} = \oplus_{n\geqslant 0}\mathrm{K}_n^{\mathrm{B}}$. Then K^{B} has a structure of a Hopf bimodule over K^{A}. The group U_I has the anti-involution θ induced by the involution $a\mapsto a^{-1}$ of $I=\mathbb{C}^*$. Let U_I^θ be the θ-fixed point set of U_I. Then $\mathscr{O}(U_I^\theta)$ is a quotient ring of $\mathscr{O}(U_I)$. The action of $\mathscr{O}(U_I)\simeq\mathbb{C}\otimes\mathrm{K}^{\mathrm{A}}$ on $\mathbb{C}\otimes\mathrm{K}^{\mathrm{B}}$, in fact, descends to the action of $\mathscr{O}(U_I^\theta)$.

We introduce $V_\theta(\lambda)$ (see § 2.3), a kind of the q-analogue of $\mathscr{O}(U_I^\theta)$. The conjecture in [EK] is then:

(i) $V_\theta(\lambda)$ has a crystal basis and a global basis.
(ii) K^{B} is isomorphic to a specialization of $V_\theta(\lambda)$ at $q=1$ as an $\mathscr{O}(U_I)$-module, and the irreducible representations correspond to the upper global basis of $V_\theta(\lambda)$ at $q=1$.

Remark. In [KM], Miemietz and the second author gave an analogous conjecture for the affine Hecke algebras of type D.

In the present paper, we prove that $V_\theta(\lambda)$ has a crystal basis and a global basis for $\mathfrak{g}=\mathfrak{gl}_\infty$ and $\lambda=0$.

More precisely, let $I=\mathbb{Z}_{\mathrm{odd}}$ be the set of odd integers. Let α_i $(i\in I)$ be the simple roots with

$$(\alpha_i,\alpha_j)=\begin{cases}2 & \text{if } i=j,\\ -1 & \text{if } i=j\pm 2,\\ 0 & \text{otherwise.}\end{cases}$$

Let θ be the involution of I given by $\theta(i)=-i$. Let $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$ be the algebra over $\mathbf{K}:=\mathbb{Q}(q)$ generated by E_i, F_i, and invertible elements T_i $(i\in I)$ satisfying the following defining relations:

(i) the T_i's commute with each other,
(ii) $T_{\theta(i)}=T_i$ for any i,

(iii) $T_iE_jT_i^{-1} = q^{(\alpha_i+\alpha_{\theta(i)},\alpha_j)}E_j$ and $T_iF_jT_i^{-1} = q^{(\alpha_i+\alpha_{\theta(i)},-\alpha_j)}F_j$ for $i, j \in I$,
(iv) $E_iF_j = q^{-(\alpha_i,\alpha_j)}F_jE_i + (\delta_{i,j} + \delta_{\theta(i),j}T_i)$ for $i, j \in I$,
(v) the E_i's and the F_i's satisfy the Serre relations (see Definition 2.1 (4)).

Then there exists a unique irreducible $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$-module $V_\theta(0)$ with a generator ϕ satisfying $E_i\phi = 0$ and $T_i\phi = \phi$ (Proposition 2.11). We define the endomorphisms $\widetilde{E}_i$ and $\widetilde{F}_i$ of $V_\theta(0)$ by

$$\widetilde{E}_i a = \sum_{n\geqslant 1} F_i^{(n-1)} a_n, \quad \widetilde{F}_i a = \sum_{n\geqslant 0} f_i^{(n+1)} a_n,$$

when writing

$$a = \sum_{n\geqslant 0} F_i^{(n)} a_n \quad \text{with } E_i a_n = 0.$$

Here $F_i^{(n)} = F_i^n/[n]!$ is the divided power. Let $\mathbf{A}_0$ be the ring of functions $a \in \mathbf{K}$ which do not have a pole at $q = 0$. Let $L_\theta(0)$ be the $\mathbf{A}_0$-submodule of $V_\theta(0)$ generated by the elements $\widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell}\phi$ ($\ell \geqslant 0$, $i_1, \ldots, i_\ell \in I$). Let $\mathrm{B}_\theta(0)$ be the subset of $L_\theta(0)/qL_\theta(0)$ consisting of the $\widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell}\phi$'s. In this paper, we prove the following theorem.

Theorem (Theorem 4.15). (i) $\widetilde{F}_i L_\theta(0) \subset L_\theta(0)$ *and* $\widetilde{E}_i L_\theta(0) \subset L_\theta(0)$,
(ii) $\mathrm{B}_\theta(0)$ *is a basis of* $L_\theta(0)/qL_\theta(0)$,
(iii) $\widetilde{F}_i\mathrm{B}_\theta(0) \subset \mathrm{B}_\theta(0)$, *and* $\widetilde{E}_i\mathrm{B}_\theta(0) \subset \mathrm{B}_\theta(0) \sqcup \{0\}$,
(iv) $\widetilde{F}_i\widetilde{E}_i(b) = b$ *for any* $b \in \mathrm{B}_\theta(0)$ *such that* $\widetilde{E}_i b \neq 0$, *and* $\widetilde{E}_i\widetilde{F}_i(b) = b$ *for any* $b \in \mathrm{B}_\theta(0)$.

Let $-$ be the bar operator of $V_\theta(0)$. Namely, $-$ is a unique endomorphism of $V_\theta(0)$ such that $\overline{\phi} = \phi$, $\overline{av} = \bar{a}\bar{v}$ and $\overline{F_i v} = F_i\bar{v}$ for $a \in \mathbf{K}$ and $v \in V_\theta(0)$. Here $\bar{a}(q) = a(q^{-1})$.

Then we prove the existence of global basis:

Theorem (Theorem 5.5). *Let* $V_\theta(0)_{\mathbf{A}}$ *be the smallest submodule of* $V_\theta(0)$ *over* $\mathbf{A}{:=}\mathbb{Q}[q, q^{-1}]$ *such that it contains* ϕ *and is stable by the* $F_i^{(n)}$*'s.*

(i) *For any* $b \in \mathrm{B}_\theta(0)$, *there exists a unique* $\mathrm{G}_\theta^{\mathrm{low}}(b) \in V_\theta(0)_{\mathbf{A}} \cap L_\theta(0)$ *such that* $\overline{\mathrm{G}_\theta^{\mathrm{low}}(b)} = \mathrm{G}_\theta^{\mathrm{low}}(b)$ *and* $b = \mathrm{G}_\theta^{\mathrm{low}}(b) \mod qL_\theta(0)$,
(ii) $L_\theta(0) = \bigoplus_{b\in\mathrm{B}_\theta(0)} \mathbf{A}_0\mathrm{G}_\theta^{\mathrm{low}}(b)$, $V_\theta(0)_{\mathbf{A}} = \bigoplus_{b\in\mathrm{B}_\theta(0)} \mathbf{A}\mathrm{G}_\theta^{\mathrm{low}}(b)$ *and* $V_\theta(0) = \bigoplus_{b\in\mathrm{B}_\theta(0)} \mathbf{K}\mathrm{G}_\theta^{\mathrm{low}}(b)$.

We call $\mathrm{G}_\theta^{\mathrm{low}}(b)$ the *lower global basis*. The $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$-module $V_\theta(0)$ has a unique symmetric bilinear form $(\,\bullet\,,\,\bullet\,)$ such that $(\phi, \phi) = 1$ and E_i and F_i are transpose to each other. The dual basis to $\{\mathrm{G}_\theta^{\mathrm{low}}(b)\}_{b\in\mathrm{B}_\theta(0)}$ with respect to $(\,\bullet\,,\,\bullet\,)$ is called an *upper global basis.*

Let us explain the strategy of our proof of these theorems. We first construct a PBW type basis $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}}$ of $V_\theta(0)$ parametrized by the θ-restricted multisegments $\mathfrak{m}$. Then, we explicitly calculate the actions of E_i and F_i in terms of the PBW basis $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}}$. Then, we prove that the PBW basis gives a crystal basis by the estimation of the coefficients of these actions. For this we use a criterion for crystal bases (Theorem 4.8).

2. General definitions and conjectures

2.1. **Quantized universal enveloping algebras and its reduced q-analogues.** We shall recall the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Let I be an index set (for simple roots), and Q the free $\mathbb{Z}$-module with a basis $\{\alpha_i\}_{i\in I}$. Let $(\,\bullet\,,\,\bullet\,)\colon Q \times Q \to \mathbb{Z}$ be a symmetric bilinear form such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}_{>0}$ for any i and $(\alpha_i^\vee, \alpha_j) \in \mathbb{Z}_{\leqslant 0}$ for $i \neq j$

where $\alpha_i^\vee := 2\alpha_i/(\alpha_i,\alpha_i)$. Let q be an indeterminate and set $\mathbf{K} := \mathbb{Q}(q)$. We define its subrings $\mathbf{A}_0$, $\mathbf{A}_\infty$ and $\mathbf{A}$ as follows.

$$\begin{aligned}\mathbf{A}_0 &= \{f \in \mathbf{K} \mid f \text{ is regular at } q=0\},\\ \mathbf{A}_\infty &= \{f \in \mathbf{K} \mid f \text{ is regular at } q=\infty\},\\ \mathbf{A} &= \mathbb{Q}[q,q^{-1}].\end{aligned}$$

Definition 2.1. *The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the $\mathbf{K}$-algebra generated by elements e_i, f_i and invertible elements t_i $(i \in I)$ with the following defining relations.*

(1) *The t_i's commute with each other.*

(2) $t_j e_i t_j^{-1} = q^{(\alpha_j,\alpha_i)} e_i$ *and* $t_j f_i t_j^{-1} = q^{-(\alpha_j,\alpha_i)} f_i$ *for any $i, j \in I$.*

(3) $[e_i, f_j] = \delta_{ij}\dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ *for $i,\ j \in I$. Here $q_i := q^{(\alpha_i,\alpha_i)/2}$.*

(4) (Serre relation) *For $i \neq j$,*

$$\sum_{k=0}^{b}(-1)^k e_i^{(k)} e_j e_i^{(b-k)} = 0,\ \sum_{k=0}^{b}(-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.$$

Here $b = 1 - (\alpha_i^\vee, \alpha_j)$ and

$$e_i^{(k)} = e_i^k/[k]_i!\,,\ f_i^{(k)} = f_i^k/[k]_i!\ ,\ [k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1})\,,\ [k]_i! = [1]_i \cdots [k]_i\,.$$

Let us denote by $U_q^-(\mathfrak{g})$ (resp. $U_q^+(\mathfrak{g})$) the $\mathbf{K}$-subalgebra of $U_q(\mathfrak{g})$ generated by the f_i's (resp. the e_i's).

Let e_i' and e_i^* be the operators on $U_q^-(\mathfrak{g})$ defined by

$$[e_i, a] = \frac{(e_i^* a)t_i - t_i^{-1}e_i' a}{q_i - q_i^{-1}} \quad (a \in U_q^-(\mathfrak{g})).$$

These operators satisfy the following formulas similar to derivations:

$$\begin{aligned}&e_i'(ab) = e_i'(a)b + (\mathrm{Ad}(t_i)a)e_i'b,\\ &e_i^*(ab) = ae_i^*b + (e_i^*a)(\mathrm{Ad}(t_i)b).\end{aligned} \tag{2.1}$$

The algebra $U_q^-(\mathfrak{g})$ has a unique symmetric bilinear form $(\bullet,\bullet)$ such that $(1,1) = 1$ and

$$(e_i'a, b) = (a, f_i b) \quad \text{for any } a, b \in U_q^-(\mathfrak{g}).$$

It is non-degenerate and satisfies $(e_i^*a, b) = (a, bf_i)$. The left multiplication of f_j, e_i' and e_i^* have the commutation relations

$$e_i' f_j = q^{-(\alpha_i,\alpha_j)} f_j e_i' + \delta_{ij},\ e_i^* f_j = f_j e_i^* + \delta_{ij}\,\mathrm{Ad}(t_i),$$

and both the e_i''s and the e_i^*'s satisfy the Serre relations.

Definition 2.2. *The reduced q-analogue $\mathcal{B}(\mathfrak{g})$ of $\mathfrak{g}$ is the $\mathbf{K}$-algebra generated by e_i' and f_i.*

2.2. **Review on crystal bases and global bases.** Since e_i' and f_i satisfy the q-boson relation, any element $a \in U_q^-(\mathfrak{g})$ can be uniquely written as

$$a = \sum_{n \geqslant 0} f_i^{(n)} a_n \quad \text{with } e_i' a_n = 0.$$

Here $f_i^{(n)} = \dfrac{f_i^n}{[n]_i!}$.

Definition 2.3. *We define the modified root operators $\widetilde{e}_i$ and $\widetilde{f}_i$ on $U_q^-(\mathfrak{g})$ by*

$$\widetilde{e}_i a = \sum_{n \geqslant 1} f_i^{(n-1)} a_n, \quad \widetilde{f}_i a = \sum_{n \geqslant 0} f_i^{(n+1)} a_n.$$

Theorem 2.4 ([K])**.** *We define*

$$\begin{aligned} L(\infty) &= \sum_{\ell \geqslant 0,\, i_1,\dots,i_\ell \in I} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \subset U_q^-(\mathfrak{g}), \\ \mathrm{B}(\infty) &= \Big\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \mod qL(\infty) \mid \ell \geqslant 0, i_1, \cdots, i_\ell \in I \Big\} \subset L(\infty)/qL(\infty). \end{aligned}$$

Then we have

(i) $\widetilde{e}_i L(\infty) \subset L(\infty)$ *and* $\widetilde{f}_i L(\infty) \subset L(\infty)$,

(ii) $\mathrm{B}(\infty)$ *is a basis of* $L(\infty)/qL(\infty)$,

(iii) $\widetilde{f}_i \mathrm{B}(\infty) \subset \mathrm{B}(\infty)$ *and* $\widetilde{e}_i \mathrm{B}(\infty) \subset \mathrm{B}(\infty) \cup \{0\}$.

We call $(L(\infty), \mathrm{B}(\infty))$ *the* crystal basis *of* $U_q^-(\mathfrak{g})$.

Let $-$ be the automorphism of $\mathbf{K}$ sending q to q^{-1}. Then $\overline{\mathbf{A}_0}$ coincides with $\mathbf{A}_\infty$.

Let V be a vector space over $\mathbf{K}$, L_0 an $\mathbf{A}_0$-submodule of V, L_∞ an $\mathbf{A}_\infty$- submodule, and $V_\mathbf{A}$ an $\mathbf{A}$-submodule. Set $E := L_0 \cap L_\infty \cap V_\mathbf{A}$.

Definition 2.5 ([K])**.** *We say that* $(L_0, L_\infty, V_\mathbf{A})$ *is* balanced *if each of* L_0*,* L_∞ *and* $V_\mathbf{A}$ *generates* V *as a* $\mathbf{K}$*-vector space, and if one of the following equivalent conditions is satisfied.*

(i) $E \to L_0/qL_0$ *is an isomorphism,*

(ii) $E \to L_\infty/q^{-1}L_\infty$ *is an isomorphism,*

(iii) $(L_0 \cap V_\mathbf{A}) \oplus (q^{-1}L_\infty \cap V_\mathbf{A}) \to V_\mathbf{A}$ *is an isomorphism,*

(iv) $\mathbf{A}_0 \otimes_\mathbb{Q} E \to L_0$, $\mathbf{A}_\infty \otimes_\mathbb{Q} E \to L_\infty$, $\mathbf{A} \otimes_\mathbb{Q} E \to V_\mathbf{A}$ *and* $\mathbf{K} \otimes_\mathbb{Q} E \to V$ *are isomorphisms.*

Let $-$ be the ring automorphism of $U_q(\mathfrak{g})$ sending q, t_i, e_i, f_i to q^{-1}, t_i^{-1}, e_i, f_i.

Let $U_q(\mathfrak{g})_\mathbf{A}$ be the $\mathbf{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$ and t_i. Similarly we define $U_q^-(\mathfrak{g})_\mathbf{A}$.

Theorem 2.6. $(L(\infty), L(\infty)^-, U_q^-(\mathfrak{g})_\mathbf{A})$ *is balanced.*

Let

$$\mathrm{G}^{\mathrm{low}} \colon L(\infty)/qL(\infty) \xrightarrow{\sim} E := L(\infty) \cap L(\infty)^- \cap U_q^-(\mathfrak{g})_\mathbf{A}$$

be the inverse of $E \xrightarrow{\sim} L(\infty)/qL(\infty)$. Then $\big\{ \mathrm{G}^{\mathrm{low}}(b) \mid b \in \mathbb{B}(\infty) \big\}$ forms a basis of $U_q^-(\mathfrak{g})$. We call it a (lower) *global basis*. It is first introduced by G. Lusztig ([L]) under the name of "canonical basis" for the A, D, E cases.

Definition 2.7. *Let*

$$\{ \mathrm{G}^{\mathrm{up}}(b) \mid b \in \mathrm{B}(\infty) \}$$

be the dual basis of $\big\{ \mathrm{G}^{\mathrm{low}}(b) \mid b \in \mathrm{B}(\infty) \big\}$ *with respect to the inner product* $(\,\bullet\,,\,\bullet\,)$*. We call it the upper global basis of* $U_q^-(\mathfrak{g})$.

2.3. **Symmetric crystals.** Let θ be an automorphism of I such that $\theta^2 = \mathrm{id}$ and $(\alpha_{\theta(i)}, \alpha_{\theta(j)}) = (\alpha_i, \alpha_j)$. Hence it extends to an automorphism of the root lattice Q by $\theta(\alpha_i) = \alpha_{\theta(i)}$, and induces an automorphism of $U_q(\mathfrak{g})$.

Definition 2.8. *Let* $\mathcal{B}_\theta(\mathfrak{g})$ *be the* $\mathbf{K}$*-algebra generated by* E_i*,* F_i*, and invertible elements* T_i $(i \in I)$ *satisfying the following defining relations:*

(i) *the* T_i*'s commute with each other,*

(ii) $T_{\theta(i)} = T_i$ *for any* i,
(iii) $T_i E_j T_i^{-1} = q^{(\alpha_i+\alpha_{\theta(i)},\alpha_j)} E_j$ *and* $T_i F_j T_i^{-1} = q^{(\alpha_i+\alpha_{\theta(i)},-\alpha_j)} F_j$ *for* $i, j \in I$,
(iv) $E_i F_j = q^{-(\alpha_i,\alpha_j)} F_j E_i + (\delta_{i,j} + \delta_{\theta(i),j} T_i)$ *for* $i, j \in I$,
(v) *the* E_i*'s and the* F_i*'s satisfy the Serre relations* (Definition 2.1 (4)).

We set $E_i^{(n)} = E_i^n/[n]_i!$ and $F_i^{(n)} = F_i^n/[n]_i!$.

Lemma 2.9. *Identifying* $U_q^-(\mathfrak{g})$ *with the subalgebra of* $\mathcal{B}_\theta(\mathfrak{g})$ *generated by the* F_i*'s, we have*

$$T_i a = \big(\mathrm{Ad}(t_i t_{\theta(i)})a\big) T_i, \tag{2.2}$$
$$E_i a = \big(\mathrm{Ad}(t_i)a\big) E_i + e'_i a + \big(\mathrm{Ad}(t_i)(e^*_{\theta(i)} a)\big) T_i \tag{2.3}$$

for $a \in U_q^-(\mathfrak{g})$.

Proof. The first relation is obvious. In order to prove the second, it is enough to show that if a satisfies (2.3), then $f_j a$ satisfies (2.3). We have

$$\begin{aligned} E_i(f_j a) &= (q^{-(\alpha_i,\alpha_j)} f_j E_i + \delta_{i,j} + \delta_{\theta(i),j} T_i) a \\ &= q^{-(\alpha_i,\alpha_j)} f_j (\big(\mathrm{Ad}(t_i)a\big) E_i + e'_i a + \big(\mathrm{Ad}(t_i)(e^*_{\theta(i)} a)\big) T_i) \\ &\qquad\qquad + \delta_{i,j} a + \delta_{\theta(i),j} \big(\mathrm{Ad}(t_i t_{\theta(i)})a\big) T_i \\ &= (\big(\mathrm{Ad}(t_i)(f_j a)\big) E_i + e'_i(f_j a) + \big(\mathrm{Ad}(t_i)(e^*_{\theta(i)}(f_j a)\big) T_i. \end{aligned}$$

Q.E.D.

The following lemma can be proved in a standard manner and we omit the proof.

Lemma 2.10. *Let* $\mathbf{K}[T_i^{\pm}; i \in I]$ *be the commutative* $\mathbf{K}$*-algebra generated by invertible elements* T_i $(i \in I)$ *with the defining relation* $T_{\theta(i)} = T_i$*. Then the map* $U_q^-(\mathfrak{g}) \otimes \mathbf{K}[T_i^{\pm}; i \in I] \otimes U_q^+(\mathfrak{g}) \to \mathcal{B}_\theta(\mathfrak{g})$ *induced by the multiplication is bijective.*

Let $\lambda \in P_+ := \{\lambda \in \mathrm{Hom}(Q, \mathbb{Q}) \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}_{\geqslant 0}$ for any $i \in I\}$ be a dominant integral weight such that $\theta(\lambda) = \lambda$.

Proposition 2.11. (i) *There exists a* $\mathcal{B}_\theta(\mathfrak{g})$*-module* $V_\theta(\lambda)$ *generated by a non-zero vector* ϕ_λ *such that*
(a) $E_i \phi_\lambda = 0$ *for any* $i \in I$,
(b) $T_i \phi_\lambda = q^{(\alpha_i,\lambda)} \phi_\lambda$ *for any* $i \in I$,
(c) $\{u \in V_\theta(\lambda) \mid E_i u = 0$ *for any* $i \in I\} = \mathbf{K}\phi_\lambda$.
Moreover such a $V_\theta(\lambda)$ *is irreducible and unique up to an isomorphism.*
(ii) *there exists a unique symmetric bilinear form* $(\bullet, \bullet)$ *on* $V_\theta(\lambda)$ *such that* $(\phi_\lambda, \phi_\lambda) = 1$ *and* $(E_i u, v) = (u, F_i v)$ *for any* $i \in I$ *and* $u, v \in V_\theta(\lambda)$*, and it is non-degenerate.*

Remark 2.12. Set $P_\theta = \{\mu \in P \mid \theta(\mu) = \mu\}$. Then $V_\theta(\lambda)$ has a weight decomposition

$$V_\theta(\lambda) = \bigoplus_{\mu \in P_\theta} V_\theta(\lambda)_\mu,$$

where $V_\theta(\lambda)_\mu = \big\{u \in V_\theta(\lambda) \mid T_i u = q^{(\alpha_i,\mu)} u\big\}$. We say that an element u of $V_\theta(\lambda)$ has a θ-weight μ and write $\mathrm{wt}_\theta(u) = \mu$ if $u \in V_\theta(\lambda)_\mu$. We have $\mathrm{wt}_\theta(E_i u) = \mathrm{wt}_\theta(u) + (\alpha_i + \alpha_{\theta(i)})$ and $\mathrm{wt}_\theta(F_i u) = \mathrm{wt}_\theta(u) - (\alpha_i + \alpha_{\theta(i)})$.

In order to prove Proposition 2.11, we shall construct two $\mathcal{B}_\theta(\mathfrak{g})$-modules.

Lemma 2.13. *Let $U_q^-(\mathfrak{g})\phi'_\lambda$ be a free $U_q^-(\mathfrak{g})$-module with a generator ϕ'_λ. Then the following action gives a structure of a $\mathcal{B}_\theta(\mathfrak{g})$-module on $U_q^-(\mathfrak{g})\phi'_\lambda$:*

$$(2.4)\quad \begin{cases} T_i(a\phi'_\lambda) &=& q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i t_{\theta(i)})a)\phi'_\lambda, \\ E_i(a\phi'_\lambda) &=& \big(e'_i a + q^{(\alpha_i,\lambda)}\,\mathrm{Ad}(t_i)(e^*_{\theta(i)}a)\big)\phi'_\lambda, \\ F_i(a\phi'_\lambda) &=& (f_i a)\phi'_\lambda, \end{cases} \qquad \text{for any } i\in I \text{ and } a\in U_q^-(\mathfrak{g}).$$

Moreover $\mathcal{B}_\theta(\mathfrak{g})/\sum_{i\in I}(\mathcal{B}_\theta(\mathfrak{g})E_i + \mathcal{B}_\theta(\mathfrak{g})(T_i - q^{(\alpha_i,\lambda)})) \to U_q^-(\mathfrak{g})\phi'_\lambda$ is an isomorphism.

Proof. We can easily check the defining relations in Definition 2.8 except the Serre relations for the E_i's. For $i \neq j \in I$, set $S = \sum_{n=0}^{b}(-1)^n E_i^{(n)}E_jE_i^{(b-n)}$ where $b = 1 - \langle h_i, \alpha_j\rangle$. It is enough to show that the action of S on $U_q^-(\mathfrak{g})\phi'_\lambda$ is equal to 0. We can check easily that $SF_k = q^{-(b\alpha_i+\alpha_j,\alpha_k)}F_kS$. Since $S\phi'_\lambda = 0$, we have $SU_q^-(\mathfrak{g})\phi'_\lambda = 0$.

Hence $U_q^-(\mathfrak{g})\phi'_\lambda$ has a $\mathcal{B}_\theta(\mathfrak{g})$-module structure.

The last statement is obvious. Q.E.D.

Lemma 2.14. *Let $U_q^-(\mathfrak{g})\phi''_\lambda$ be a free $U_q^-(\mathfrak{g})$-module with a generator ϕ''_λ. Then the following action gives a structure of a $\mathcal{B}_\theta(\mathfrak{g})$-module on $U_q^-(\mathfrak{g})\phi''_\lambda$:*

$$(2.5)\quad \begin{cases} T_i(a\phi''_\lambda) &=& q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i t_{\theta(i)})a)\phi''_\lambda, \\ E_i(a\phi''_\lambda) &=& (e'_i a)\phi''_\lambda, \\ F_i(a\phi''_\lambda) &=& \big(f_i a + q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i)a)f_{\theta(i)}\big)\phi''_\lambda, \end{cases} \qquad \text{for any } i\in I \text{ and } a\in U_q^-(\mathfrak{g}).$$

Moreover, there exists a non-degenerate bilinear form $\langle\,\bullet\,,\,\bullet\,\rangle\colon U_q^-(\mathfrak{g})\phi'_\lambda \times U_q^-(\mathfrak{g})\phi''_\lambda \to \mathbf{K}$ such that $\langle F_iu, v\rangle = \langle u, E_iv\rangle$, $\langle E_iu, v\rangle = \langle u, F_iv\rangle$, $\langle T_iu, v\rangle = \langle u, T_iv\rangle$ for $u \in U_q^-(\mathfrak{g})\phi'_\lambda$ and $v \in U_q^-(\mathfrak{g})\phi''_\lambda$, and $\langle \phi'_\lambda, \phi''_\lambda\rangle = 1$.

Proof. There exists a unique symmetric bilinear form $(\,\bullet\,,\,\bullet\,)$ on $U_q^-(\mathfrak{g})$ such that $(1,1) = 1$ and f_i and e'_i are transpose to each other. Let us define $\langle\,\bullet\,,\,\bullet\,\rangle\colon U_q^-(\mathfrak{g})\phi'_\lambda \times U_q^-(\mathfrak{g})\phi''_\lambda \to \mathbf{K}$ by $\langle a\phi'_\lambda, b\phi''_\lambda\rangle = (a,b)$ for $a \in U_q^-(\mathfrak{g})$ and $b \in U_q^-(\mathfrak{g})$. Then we can easily check $\langle F_iu, v\rangle = \langle u, E_iv\rangle$, $\langle T_iu, v\rangle = \langle u, T_iv\rangle$. Since e^*_i is transpose to the right multiplication of f_i, we have $\langle E_iu, v\rangle = \langle u, F_iv\rangle$. Hence the action of E_i, F_i, T_i on $U_q^-(\mathfrak{g})\phi''_\lambda$ satisfy the defining relations in Definition 2.8. Q.E.D.

Proof of Proposition 2.11. Since $E_i\phi''_\lambda = 0$ and ϕ''_λ has a θ-weight λ, there exists a unique $\mathcal{B}_\theta(\mathfrak{g})$-linear morphism $\psi\colon U_q^-(\mathfrak{g})\phi'_\lambda \to U_q^-(\mathfrak{g})\phi''_\lambda$ sending ϕ'_λ to ϕ''_λ. Let $V_\theta(\lambda)$ be its image $\psi(U_q^-(\mathfrak{g})\phi'_\lambda)$.

(i)(c) follows from $\{u \in U_q^-(\mathfrak{g}) \mid e'_iu = 0 \text{ for any } i\} = \mathbf{K}$ applying to $U_q^-(\mathfrak{g})\phi''_\lambda \supset V_\theta(\lambda)$. The other properties (a), (b) are obvious. Let us show that $V_\theta(\lambda)$ is irreducible. Let S be a non-zero $\mathcal{B}_\theta(\mathfrak{g})$-submodule. Then S contains a non-zero vector v such that $E_iv = 0$ for any i. Then (c) implies that v is a constant multiple of ϕ_λ. Hence $S = V_\theta(\lambda)$.

Let us prove (ii). For $u, u' \in U_q^-(\mathfrak{g})\phi'_\lambda$, set $(\!(u, u')\!) = \langle u, \psi(u')\rangle$. Then it is a bilinear form on $U_q^-(\mathfrak{g})\phi'_\lambda$ which satisfies

$$(2.6)\quad (\!(\phi'_\lambda, \phi'_\lambda)\!) = 1,\ (\!(F_iu, u')\!) = (\!(u, E_iu')\!),\ (\!(E_iu, u')\!) = (\!(u, F_iu')\!),\ (\!(T_iu, u')\!) = (\!(u, T_iu')\!).$$

It is easy to see that a bilinear form which satisfies (2.6) is unique. Since $(\!(u', u)\!)$ also satisfies (2.6), $(\!(u, u')\!)$ is a symmetric bilinear form on $U_q^-(\mathfrak{g})\phi'_\lambda$. Since $\psi(u') = 0$ implies $(\!(u, u')\!) = 0$, $(\!(u, u')\!)$ induces a symmetric bilinear form on $V_\theta(\lambda)$. Since $(\,\bullet\,,\,\bullet\,)$ is non-degenerate on $U_q^-(\mathfrak{g})$, $(\!(\,\bullet\,,\,\bullet\,)\!)$ is a non-degenerate symmetric bilinear form on $V_\theta(\lambda)$.

Q.E.D.

Lemma 2.15. *There exists a unique endomorphism* $-$ *of* $V_\theta(\lambda)$ *such that* $\overline{\phi_\lambda} = \phi_\lambda$ *and* $\overline{av} = \bar{a}\bar{v}$, $\overline{F_i v} = F_i \bar{v}$ *for any* $a \in \mathbf{K}$ *and* $v \in V_\theta(\lambda)$.

Proof. The uniqueness is obvious.

Let ξ be an anti-involution of $U_q^-(\mathfrak{g})$ such that $\xi(q) = q^{-1}$ and $\xi(f_i) = f_{\theta(i)}$. Let $\tilde{\rho}$ be an element of $\mathbb{Q} \otimes P$ such that $(\tilde{\rho}, \alpha_i) = (\alpha_i, \alpha_{\theta(i)})/2$. Define $c(\mu) = \big((\mu + \tilde{\rho}, \theta(\mu + \tilde{\rho})) - (\tilde{\rho}, \theta(\tilde{\rho}))\big)/2 + (\lambda, \mu)$ for $\mu \in P$. Then it satisfies

$$c(\mu) - c(\mu - \alpha_i) = (\lambda + \mu, \alpha_{\theta(i)}).$$

Then we define the endomorphism Φ of $U_q^-(\mathfrak{g})\phi''_\lambda$ by $\Phi(a\phi''_\lambda) = q^{-c(\mu)}\xi(a)\phi''_\lambda$ for $a \in U_q^-(\mathfrak{g})_\mu$. Let us show that

$$\Phi(F_i(a\phi''_\lambda)) = F_i\Phi(a\phi''_\lambda) \quad \text{for any } a \in U_q^-(\mathfrak{g}). \tag{2.7}$$

For $a \in U_q^-(\mathfrak{g})_\mu$, we have

$$\begin{aligned} \Phi(F_i(a\phi''_\lambda)) &= \Phi\big(f_i a + q^{(\alpha_i, \lambda+\mu)} a f_{\theta(i)}\big)\phi''_\lambda \\ &= \big(q^{-c(\mu-\alpha_i)}\xi(a) f_{\theta(i)} + q^{-(\alpha_i,\lambda+\mu) - c(\mu - \alpha_{\theta(i)})} f_i \xi(a)\big)\phi''_\lambda. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} F_i\Phi(a\phi''_\lambda) &= F_i\big(q^{-c(\mu)}\xi(a)\phi''_\lambda\big) \\ &= q^{-c(\mu)}\big(f_i\xi(a) + q^{(\alpha_i, \lambda+\theta(\mu))}\xi(a) f_{\theta(i)}\big)\phi''_\lambda. \end{aligned}$$

Therefore we obtain (2.7).

Hence Φ induces the desired endomorphism of $V_\theta(0) \subset U_q^-(\mathfrak{g})\phi''_\lambda$. Q.E.D.

Hereafter we assume further that

$$\text{there is no } i \in I \text{ such that } \theta(i) = i.$$

We conjecture that $V_\theta(\lambda)$ has a crystal basis. This means the following. Since E_i and F_i satisfy the q-boson relation, any $u \in V_\theta(\lambda)$ can be uniquely written as $u = \sum_{n \geqslant 0} F_i^{(n)} u_n$ with $E_i u_n = 0$. We define the modified root operators $\widetilde{E}_i$ and $\widetilde{F}_i$ by:

$$\widetilde{E}_i(u) = \sum_{n \geqslant 1} F_i^{(n-1)} u_n \text{ and } \widetilde{F}_i(u) = \sum_{n \geqslant 0} F_i^{(n+1)} u_n.$$

Let $L_\theta(\lambda)$ be the $\mathbf{A}_0$-submodule of $V_\theta(\lambda)$ generated by $\widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell}\phi_\lambda$ ($\ell \geqslant 0$ and $i_1, \ldots, i_\ell \in I$), and let $\mathrm{B}_\theta(\lambda)$ be the subset

$$\Big\{\widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell}\phi_\lambda \bmod qL_\theta(\lambda) \mid \ell \geqslant 0,\, i_1, \ldots, i_\ell \in I\Big\}$$

of $L_\theta(\lambda)/qL_\theta(\lambda)$.

Conjecture 2.16. Let λ be a dominant integral weight such that $\theta(\lambda) = \lambda$. Then we have

(1) $\widetilde{F}_i L_\theta(\lambda) \subset L_\theta(\lambda)$ and $\widetilde{E}_i L_\theta(\lambda) \subset L_\theta(\lambda)$,
(2) $\mathrm{B}_\theta(\lambda)$ is a basis of $L_\theta(\lambda)/qL_\theta(\lambda)$,
(3) $\widetilde{F}_i \mathrm{B}_\theta(\lambda) \subset \mathrm{B}_\theta(\lambda)$, and $\widetilde{E}_i \mathrm{B}_\theta(\lambda) \subset \mathrm{B}_\theta(\lambda) \sqcup \{0\}$,
(4) $\widetilde{F}_i\widetilde{E}_i(b) = b$ for any $b \in \mathrm{B}_\theta(\lambda)$ such that $\widetilde{E}_i b \neq 0$, and $\widetilde{E}_i\widetilde{F}_i(b) = b$ for any $b \in \mathrm{B}_\theta(\lambda)$.

As in [K], we have

Lemma 2.17. *Assume* Conjecture 2.16. *Then we have*

(i) $L_\theta(\lambda) = \{v \in V_\theta(\lambda) \mid (L_\theta(\lambda), v) \subset \mathbf{A}_0\}$,

(ii) *Let* $(\bullet,\bullet)_0$ *be the* $\mathbb{Q}$*-valued symmetric bilinear form on* $L_\theta(\lambda)/qL_\theta(\lambda)$ *induced by* $(\bullet,\bullet)$. *Then* $\mathrm{B}_\theta(\lambda)$ *is an orthonormal basis with respect to* $(\bullet,\bullet)_0$.

Moreover we conjecture that $V_\theta(\lambda)$ has a global crystal basis. Namely we have

Conjecture 2.18. $(L_\theta(\lambda), L_\theta(\lambda)^-, V_\theta(\lambda)^{\mathrm{low}}_{\mathbf{A}})$ is balanced. Here $V_\theta(\lambda)^{\mathrm{low}}_{\mathbf{A}} := U_q^-(\mathfrak{g})_{\mathbf{A}}\phi_\lambda$.

Its dual version is as follows.

Let us denote by $V_\theta(\lambda)^{\mathrm{up}}_{\mathbf{A}}$ the dual space $\{v \in V_\theta(\lambda) \mid (V_\theta(\lambda)^{\mathrm{low}}_{\mathbf{A}}, v) \subset \mathbf{A}\}$. Then Conjecture 2.18 is equivalent to the following conjecture.

Conjecture 2.19. $(L_\theta(\lambda), c(L_\theta(\lambda)), V_\theta(\lambda)^{\mathrm{up}}_{\mathbf{A}})$ is balanced.

Here c is a unique endomorphism of $V_\theta(\lambda)$ such that $c(\phi_\lambda) = \phi_\lambda$ and $c(av) = \bar{a}c(v)$, $c(E_i v) = E_i c(v)$ for any $a \in \mathbf{K}$ and $v \in V_\theta(\lambda)$. We have $(c(v'), v) = \overline{(v', \bar{v})}$ for any $v, v' \in V_\theta(\lambda)$.

Note that $V_\theta(\lambda)^{\mathrm{up}}_{\mathbf{A}}$ is the largest $\mathbf{A}$-submodule M of $V_\theta(\lambda)$ such that M is invariant by the $E_i^{(n)}$'s and $M \cap \mathbf{K}\phi_\lambda = \mathbf{A}\phi_\lambda$.

By Conjecture 2.19, $L_\theta(\lambda) \cap c(L_\theta(\lambda)) \cap V_\theta(0)^{\mathrm{up}}_{\mathbf{A}} \to L_\theta(\lambda)/qL_\theta(\lambda)$ is an isomorphism. Let $\mathrm{G}^{\mathrm{up}}_\theta$ be its inverse. Then $\{\mathrm{G}^{\mathrm{up}}_\theta(b)\}_{b\in \mathrm{B}_\theta(\lambda)}$ is a basis of $V_\theta(\lambda)$, which we call the *upper global basis* of $V_\theta(\lambda)$. Note that $\{\mathrm{G}^{\mathrm{up}}_\theta(b)\}_{b\in \mathrm{B}_\theta(\lambda)}$ is the dual basis to $\{\mathrm{G}^{\mathrm{low}}_\theta(b)\}_{b\in \mathrm{B}_\theta(\lambda)}$ with respect to the inner product of $V_\theta(\lambda)$.

We shall prove these conjectures in the case $\mathfrak{g} = \mathfrak{gl}_\infty$ and $\lambda = 0$.

3. PBW basis of $V_\theta(0)$ for $\mathfrak{g} = \mathfrak{gl}_\infty$

3.1. **Review on the PBW basis.** In the sequel, we set $I = \mathbb{Z}_{\mathrm{odd}}$ and

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{for } i = j, \\ -1 & \text{for } j = i \pm 2, \\ 0 & \text{otherwise,} \end{cases}$$

and we consider the corresponding quantum group $U_q(\mathfrak{gl}_\infty)$. In this case, we have $q_i = q$. We write $[n]$ and $[n]!$ for $[n]_i$ and $[n]_i!$ for short.

We can parametrize the crystal basis $\mathrm{B}(\infty)$ by the multisegments. We shall recall this parametrization and the PBW basis.

Definition 3.1. *For* $i, j \in I$ *such that* $i \leqslant j$, *we define a segment* $\langle i, j\rangle$ *as the interval* $[i, j] \subset I := \mathbb{Z}_{odd}$. *A multisegment is a formal finite sum of segments:*

$$\mathfrak{m} = \sum_{i\leqslant j} m_{ij}\langle i, j\rangle$$

with $m_{i,j} \in \mathbb{Z}_{\geqslant 0}$. *We call* m_{ij} *the multiplicity of a segment* $\langle i, j\rangle$. *If* $m_{i,j} > 0$, *we sometimes say that* $\langle i, j\rangle$ *appears in* $\mathfrak{m}$. *We sometimes write* $m_{i,j}(\mathfrak{m})$ *for* $m_{i,j}$. *We write sometimes* $\langle i\rangle$ *for* $\langle i, i\rangle$. *We denote by* $\mathcal{M}$ *the set of multisegments. We denote by* $\emptyset$ *the zero element* (*or the empty multisegment*) *of* $\mathcal{M}$.

Definition 3.2. *For two segments* $\langle i_1, j_1\rangle$ *and* $\langle i_2, j_2\rangle$, *we define the ordering* $\geqslant_{PBW}$ *by the following:*

$$\langle i_1, j_1\rangle \geqslant_{PBW} \langle i_2, j_2\rangle \iff \begin{cases} j_1 > j_2 \\ \text{or} \\ j_1 = j_2 \text{ and } i_1 \geqslant i_2. \end{cases}$$

We call this ordering the PBW-ordering.

Definition 3.3. *For a multisegment* $\mathfrak{m}$, *we define the element* $P(\mathfrak{m}) \in U_q^-(\mathfrak{gl}_\infty)$ *as follows.*

(1) *For a segment* $\langle i,j\rangle$, *we define the element* $\langle i,j\rangle \in U_q^-(\mathfrak{gl}_\infty)$ *inductively by*

$$\begin{aligned}\langle i,i\rangle &= f_i,\\ \langle i,j\rangle &= \langle i,j-2\rangle\langle j,j\rangle - q\langle j,j\rangle\langle i,j-2\rangle \quad \text{for } i<j.\end{aligned}$$

(2) *For a multisegment* $\mathfrak{m} = \sum_{i\leqslant j} m_{ij}\langle i,j\rangle$, *we define*

$$P(\mathfrak{m}) = \overrightarrow{\prod} \langle i,j\rangle^{(m_{ij})}.$$

Here the product $\overrightarrow{\prod}$ *is taken over segments appearing in* $\mathfrak{m}$ *from large to small with respect to the PBW-ordering. The element* $\langle i,j\rangle^{(m_{ij})}$ *is the divided power of* $\langle i,j\rangle$ *i.e.*

$$\langle i,j\rangle^{(n)} = \begin{cases} \dfrac{1}{[n]!}\langle i,j\rangle^n & \text{for } n>0,\\ 1 & \text{for } n=0,\\ 0 & \text{for } n<0.\end{cases}$$

Hence the weight of $P(\mathfrak{m})$ is equal to $\mathrm{wt}(\mathfrak{m}) := -\sum_{i\leqslant k\leqslant j} m_{i,j}\alpha_k$: $t_iP(\mathfrak{m})t_i^{-1} = q^{(\alpha_i,\mathrm{wt}(\mathfrak{m}))}P(\mathfrak{m})$.

Theorem 3.4 ([L]). *The set of elements* $\{P(\mathfrak{m}) \mid \mathfrak{m}\in\mathcal{M}\}$ *is a* $\mathbf{K}$*-basis of* $U_q^-(\mathfrak{gl}_\infty)$. *Moreover this is an* $\mathbf{A}$*-basis of* $U_q^-(\mathfrak{gl}_\infty)_{\mathbf{A}}$. *We call this basis the* PBW basis *of* $U_q^-(\mathfrak{gl}_\infty)$.

Definition 3.5. *For two segments* $\langle i_1,j_1\rangle$ *and* $\langle i_2,j_2\rangle$, *we define the ordering* $\geqslant_{cry}$ *by the following:*

$$\langle i_1,j_1\rangle \geqslant_{cry} \langle i_2,j_2\rangle \Longleftrightarrow \begin{cases} j_1>j_2\\ \text{or}\\ j_1=j_2 \text{ and } i_1\leqslant i_2.\end{cases}$$

We call this ordering the crystal ordering.

Example 3.6. The crystal ordering is different from the PBW-ordering. For example, we have $\langle -1,1\rangle >_{\mathrm{cry}} \langle 1,1\rangle >_{\mathrm{cry}} \langle -1,-1\rangle$, while we have $\langle 1,1\rangle >_{\mathrm{PBW}} \langle -1,1\rangle >_{\mathrm{PBW}} \langle -1,-1\rangle$.

Definition 3.7. *We define the crystal structure on* $\mathcal{M}$ *as follows: for* $\mathfrak{m} = \sum m_{i,j}\langle i,j\rangle \in \mathcal{M}$ *and* $i\in I$, *set* $A_k^{(i)}(\mathfrak{m}) = \sum_{k'\geqslant k}(m_{i,k'} - m_{i+2,k'+2})$ *for* $k\geqslant i$. *Define* $\varepsilon_i(\mathfrak{m})$ *as* $\max\left\{A_k^{(i)}(\mathfrak{m}) \mid k\geqslant i\right\} \geqslant 0$.

(i) *If* $\varepsilon_i(\mathfrak{m})=0$, *then define* $\tilde{e}_i(\mathfrak{m})=0$. *If* $\varepsilon_i(\mathfrak{m})>0$, *let* k_e *be the largest* $k\geqslant i$ *such that* $\varepsilon_i(\mathfrak{m}) = A_k^{(i)}(\mathfrak{m})$ *and define* $\tilde{e}_i(\mathfrak{m}) = \mathfrak{m} - \langle i,k_e\rangle + \delta_{k_e\neq i}\langle i+2,k_e\rangle$.

(ii) *Let* k_f *be the smallest* $k\geqslant i$ *such that* $\varepsilon_i(\mathfrak{m}) = A_k^{(i)}(\mathfrak{m})$ *and define* $\tilde{f}_i(\mathfrak{m}) = \mathfrak{m} - \delta_{k_f\neq i}\langle i+2,k_f\rangle + \langle i,k_f\rangle$.

Remark 3.8. For $i\in I$, the actions of the operators $\widetilde{e}_i$ and $\widetilde{f}_i$ on $\mathfrak{m}\in\mathcal{M}$ are also described by the following algorithm:

Step 1. Arrange the segments in $\mathfrak{m}$ in the crystal ordering.
Step 2. For each segment $\langle i,j\rangle$, write $-$, and for each segment $\langle i+2,j\rangle$, write $+$.
Step 3. In the resulting sequence of $+$ and $-$, delete a subsequence of the form $+-$ and keep on deleting until no such subsequence remains.

Then we obtain a sequence of the form $--\cdots-++\cdots+$.

(1) $\varepsilon_i(\mathfrak{m})$ is the total number of $-$ in the resulting sequence.
(2) $\widetilde{f}_i(\mathfrak{m})$ is given as follows:

(a) if the leftmost $+$ corresponds to a segment $\langle i+2, j\rangle$, then replace it with $\langle i, j\rangle$,
(b) if no $+$ exists, add a segment $\langle i, i\rangle$ to $\mathfrak{m}$.

(3) $\widetilde{e}_i(\mathfrak{m})$ is given as follows:
(a) if the rightmost $-$ corresponds to a segment $\langle i, j\rangle$, then replace it with $\langle i+2, j\rangle$,
(b) if no $-$ exists, then $\widetilde{e}_i(\mathfrak{m}) = 0$.

Let us introduce a linear ordering on the set $\mathcal{M}$ of multisegments, lexicographic with respect to the crystal ordering on the set of segments.

Definition 3.9. *For $\mathfrak{m} = \sum_{i \leqslant j} m_{i,j}\langle i, j\rangle \in \mathcal{M}$ and and $\mathfrak{m}' = \sum_{i \leqslant j} m'_{i,j}\langle i, j\rangle \in \mathcal{M}$, we define $\mathfrak{m}' \underset{\text{cry}}{<} \mathfrak{m}$ if there exist $i_0 \leqslant j_0$ such that $m'_{i_0,j_0} < m_{i_0,j_0}$, $m'_{i,j_0} = m_{i,j_0}$ for $i < i_0$, and $m'_{i,j} = m_{i,j}$ for $j > j_0$ and $i \leqslant j$.*

Theorem 3.10. (i) $L(\infty) = \bigoplus_{\mathfrak{m} \in \mathcal{M}} \mathbf{A}_0 P(\mathfrak{m})$.

(ii) $\mathrm{B}(\infty) = \{P(\mathfrak{m}) \bmod qL(\infty) \mid \mathfrak{m} \in \mathcal{M}\}$.

(iii) *We have*

$$\begin{aligned} \widetilde{e}_i P(\mathfrak{m}) &\equiv P(\widetilde{e}_i(\mathfrak{m})) \quad \bmod qL(\infty), \\ \widetilde{f}_i P(\mathfrak{m}) &\equiv P(\widetilde{f}_i(\mathfrak{m})) \quad \bmod qL(\infty). \end{aligned}$$

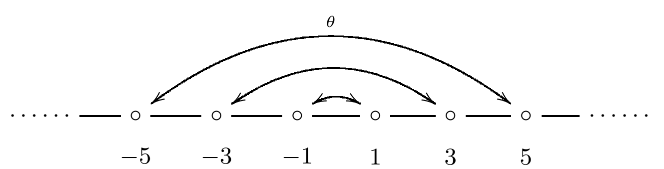
Note that $\widetilde{e}_i$ and $\widetilde{f}_i$ in the left-hand-side is the modified root operators.

(iv) *We have*

$$\overline{P(\mathfrak{m})} \in P(\mathfrak{m}) + \sum_{\mathfrak{m}' \underset{\text{cry}}{<} \mathfrak{m}} \mathbf{A} P(\mathfrak{m}').$$

Therefore we can index the crystal basis by multisegments. By this theorem we can easily see by a standard argument that $(L(\infty), L(\infty)^-, U_q^-(\mathfrak{g})_{\mathbf{A}})$ is balanced, and there exists a unique $\mathrm{G}^{\text{low}}(\mathfrak{m}) \in L(\infty) \cap U_q^-(\mathfrak{g})_{\mathbf{A}}$ such that $\mathrm{G}^{\text{low}}(\mathfrak{m})^- = \mathrm{G}^{\text{low}}(\mathfrak{m})$ and $\mathrm{G}^{\text{low}}(\mathfrak{m}) \equiv P(\mathfrak{m}) \bmod qL(\infty)$. The basis $\{\mathrm{G}^{\text{low}}(\mathfrak{m})\}_{\mathfrak{m} \in \mathcal{M}}$ is a lower global basis.

3.2. **θ-restricted multisegments.** We consider the Dynkin diagram involution θ of I defined by $\theta(i) = -i$ for $i \in I = \mathbb{Z}_{\text{odd}}$.



We shall prove in this case Conjectures 2.16 and 2.18 for $\lambda = 0$ (Theorems 4.15 and 5.5).

We set

$$\begin{aligned} \widetilde{V}_\theta(0) &:= \mathcal{B}_\theta(\mathfrak{gl}_\infty) \Big/ \sum_{i \in I} \big(\mathcal{B}_\theta(\mathfrak{gl}_\infty) E_i + \mathcal{B}_\theta(\mathfrak{gl}_\infty)(T_i - 1) + \mathcal{B}_\theta(\mathfrak{gl}_\infty)(F_i - F_{\theta(i)})\big) \\ &\simeq U_q^-(\mathfrak{gl}_\infty) \Big/ \sum_i U_q^-(\mathfrak{gl}_\infty)(f_i - f_{\theta(i)}). \end{aligned}$$

Let $\widetilde{\phi}$ be the generator of $\widetilde{V}_\theta(0)$ corresponding to $1 \in \mathcal{B}_\theta(\mathfrak{gl}_\infty)$. Since $F_i \phi''_0 = (f_i + f_{\theta(i)})\phi''_0 = F_{\theta(i)}\phi''_0$, we have an epimorphism of $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$-modules

$$\widetilde{V}_\theta(0) \twoheadrightarrow V_\theta(0). \tag{3.1}$$

We shall see later that it is in fact an isomorphism (see Theorem 4.15).

Definition 3.11. *If a multisegment* $\mathfrak{m}$ *has the form*

$$\mathfrak{m} = \sum_{-j \leqslant i \leqslant j} m_{ij} \langle i, j \rangle,$$

we call $\mathfrak{m}$ *a* θ-restricted *multisegment. We denote by* $\mathcal{M}_\theta$ *the set of* θ*-restricted multisegments.*

Definition 3.12. *For a* θ*-restricted segment* $\langle i, j \rangle$*, we define its modified divided power by*

$$\langle i, j \rangle^{[m]} = \begin{cases} \langle i, j \rangle^{(m)} = \dfrac{1}{[m]!} \langle i, j \rangle^m & (i \neq -j), \\ \dfrac{1}{\prod_{\nu=1}^m [2\nu]} \langle -j, j \rangle^m & (i = -j). \end{cases}$$

We understand that $\langle i, j \rangle^{[m]}$ is equal to 1 for $m = 0$ and vanishes for $m < 0$.

Definition 3.13. *For* $\mathfrak{m} \in \mathcal{M}_\theta$*, we define the element* $P_\theta(\mathfrak{m}) \in U_q^-(\mathfrak{gl}_\infty) \subset \mathcal{B}_\theta(\mathfrak{gl}_\infty)$ *by*

$$P_\theta(\mathfrak{m}) = \overrightarrow{\prod_{\langle i,j \rangle \in \mathfrak{m}}} \langle i, j \rangle^{[m_{ij}]}.$$

Here the product $\overrightarrow{\prod}$ *is taken over the segments appearing in* $\mathfrak{m}$ *from large to small with respect to the PBW-ordering.*

If an element $\mathfrak{m}$ *of the free abelian group generated by* $\langle i, j \rangle$ *does not belong to* $\mathcal{M}_\theta$*, we understand* $P_\theta(\mathfrak{m}) = 0$*.*

We will prove later that $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m} \in \mathcal{M}_\theta}$ is a basis of $V_\theta(0)$ (see Theorem 4.15). Here and hereafter, we write ϕ instead of $\phi_0 \in V_\theta(0)$.

3.3. **Commutation relations of $\langle i, j \rangle$.** In the sequel, we regard $U_q^-(\mathfrak{gl}_\infty)$ as a subalgebra of $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$ by $f_i \mapsto F_i$.

We shall give formulas to express products of segments by a PBW basis.

Proposition 3.14. *For* $i, j, k, l \in I$*, we have*
(1) $\langle i, j \rangle \langle k, \ell \rangle = \langle k, \ell \rangle \langle i, j \rangle$ *for* $i \leqslant j$*,* $k \leqslant \ell$ *and* $j < k - 2$*,*
(2) $\langle i, j \rangle \langle j + 2, k \rangle = \langle i, k \rangle + q \langle j + 2, k \rangle \langle i, j \rangle$ *for* $i \leqslant j < k$*,*
(3) $\langle j, k \rangle \langle i, \ell \rangle = \langle i, \ell \rangle \langle j, k \rangle$ *for* $i < j \leqslant k < \ell$*,*
(4) $\langle i, k \rangle \langle j, k \rangle = q^{-1} \langle j, k \rangle \langle i, k \rangle$ *for* $i < j \leqslant k$*,*
(5) $\langle i, j \rangle \langle i, k \rangle = q^{-1} \langle i, k \rangle \langle i, j \rangle$ *for* $i \leqslant j < k$*,*
(6) $\langle i, k \rangle \langle j, \ell \rangle = \langle j, \ell \rangle \langle i, k \rangle + (q^{-1} - q) \langle i, \ell \rangle \langle j, k \rangle$ *for* $i < j \leqslant k < \ell$*.*

Proof. (1) is obvious. We prove (2) by the induction on $k - j$. If $k - j = 2$, it is trivial by the definition. If $j < k - 2$, then $\langle k \rangle$ and $\langle i, j \rangle$ commute. Thus, we have

$$\begin{aligned} \langle i, j \rangle \langle j + 2, k \rangle &= \langle i, j \rangle \big(\langle j + 2, k - 2 \rangle \langle k \rangle - q \langle k \rangle \langle j + 2, k - 2 \rangle \big) \\ &= \big(\langle i, k - 2 \rangle + q \langle j + 2, k - 2 \rangle \langle i, j \rangle \big) \langle k \rangle - q \langle k \rangle \langle i, j \rangle \langle j + 2, k - 2 \rangle \\ &= \langle i, k - 2 \rangle \langle k \rangle + q \langle j + 2, k - 2 \rangle \langle k \rangle \langle i, j \rangle \\ &\qquad - q \langle k \rangle \big(\langle i, k - 2 \rangle + q \langle j + 2, k - 2 \rangle \langle i, j \rangle \big) \\ &= \langle i, k \rangle + \langle j + 2, k \rangle \langle i, j \rangle. \end{aligned}$$

In order to prove the other relations, we first show the following special cases.

Lemma 3.15. *We have for any* $j \in I$
(a) $\langle j - 2, j \rangle \langle j \rangle = q^{-1} \langle j \rangle \langle j - 2, j \rangle$ *and* $\langle j \rangle \langle j, j + 2 \rangle = q^{-1} \langle j, j + 2 \rangle \langle j \rangle$*,*

(b) $\langle j\rangle\langle j-2,j+2\rangle=\langle j-2,j+2\rangle\langle j\rangle$,
(c) $\langle j-2,j\rangle\langle j,j+2\rangle=\langle j,j+2\rangle\langle j-2,j\rangle+(q^{-1}-q)\langle j-2,j+2\rangle\langle j\rangle$.

Proof. The first equality in (a) follows from

$$\begin{aligned}\langle j-2,j\rangle\langle j\rangle-q^{-1}\langle j\rangle\langle j-2,j\rangle &= \big(f_{j-2}f_j-qf_jf_{j-2}\big)f_j-q^{-1}f_j\big(f_{j-2}f_j-qf_jf_{j-2}\big)\\ &= f_{j-2}f_j^2-(q+q^{-1})f_jf_{j-2}f_j+f_j^2f_{j-2}=0.\end{aligned}$$

We can similarly prove the second.

Let us show (b) and (c). We have, by (a)

$$\begin{aligned}&\langle j-2,j\rangle\langle j,j+2\rangle=\langle j-2,j\rangle\big(\langle j\rangle\langle j+2\rangle-q\langle j+2\rangle\langle j\rangle\big)\\ &\quad=q^{-1}\langle j\rangle\langle j-2,j\rangle\langle j+2\rangle-q\big(\langle j-2,j+2\rangle+q\langle j+2\rangle\langle j-2,j\rangle\big)\langle j\rangle\\ &\quad=q^{-1}\langle j\rangle\big(\langle j-2,j+2\rangle+q\langle j+2\rangle\langle j-2,j\rangle\big)\\ &\qquad\quad-q\langle j-2,j+2\rangle\langle j\rangle-q\langle j+2\rangle\langle j\rangle\langle j-2,j\rangle\\ &\quad=\big(\langle j\rangle\langle j+2\rangle-q\langle j+2\rangle\langle j\rangle\big)\langle j-2,j\rangle+q^{-1}\langle j\rangle\langle j-2,j+2\rangle-q\langle j-2,j+2\rangle\langle j\rangle\\ &\quad=\langle j,j+2\rangle\langle j-2,j\rangle+q^{-1}\langle j\rangle\langle j-2,j+2\rangle-q\langle j-2,j+2\rangle\langle j\rangle.\end{aligned}\tag{3.2}$$

Similarly, we have

$$\begin{aligned}&\langle j-2,j\rangle\langle j,j+2\rangle=\big(\langle j-2\rangle\langle j\rangle-q\langle j\rangle\langle j-2\rangle\big)\langle j,j+2\rangle\\ &\quad=q^{-1}\langle j-2\rangle\langle j,j+2\rangle\langle j\rangle-q\langle j\rangle\big(\langle j-2,j+2\rangle+q\langle j,j+2\rangle\langle j-2\rangle\big)\\ &\quad=q^{-1}\big(\langle j-2,j+2\rangle+q\langle j,j+2\rangle\langle j-2\rangle\big)\langle j\rangle\\ &\qquad\quad-q\langle j\rangle\langle j-2,j+2\rangle-q\langle j,j+2\rangle\langle j\rangle\langle j-2\rangle\\ &\quad=\langle j,j+2\rangle\big(\langle j-2\rangle\langle j\rangle-q\langle j\rangle\langle j-2\rangle\big)+q^{-1}\langle j-2,j+2\rangle\langle j\rangle-q\langle j\rangle\langle j-2,j+2\rangle\\ &\quad=\langle j,j+2\rangle\langle j-2,j\rangle+q^{-1}\langle j-2,j+2\rangle\langle j\rangle-q\langle j\rangle\langle j-2,j+2\rangle.\end{aligned}\tag{3.3}$$

Then, (3.2) and (3.3) imply (b) and (c). Q.E.D.

We shall resume the proof of Proposition 3.14. By Lemma 3.15 (b), $\langle i,k\rangle$ commutes with $\langle j\rangle$ for $i<j<k$. Thus we obtain (3).

We shall show (4) by the induction on $k-j$. Suppose $k-j=0$. The case $i=k-2$ is nothing but Lemma 3.15 (a).

If $i<k-2$, then

$$\begin{aligned}\langle i,k\rangle\langle k\rangle &= \langle i,k-4\rangle\langle k-2,k\rangle\langle k\rangle-q\langle k-2,k\rangle\langle i,k-4\rangle\langle k\rangle\\ &= q^{-1}\langle k\rangle\langle i,k-4\rangle\langle k-2,k\rangle-\langle k\rangle\langle k-2,k\rangle\langle i,k-4\rangle=q^{-1}\langle k\rangle\langle i,k\rangle.\end{aligned}$$

Suppose $k-j>0$. By using the induction hypothesis and (3), we have

$$\begin{aligned}\langle i,k\rangle\langle j,k\rangle &= \langle i,k\rangle\langle j\rangle\langle j+2,k\rangle-q\langle i,k\rangle\langle j+2,k\rangle\langle j\rangle\\ &= \langle j\rangle\langle i,k\rangle\langle j+2,k\rangle-\langle j+2,k\rangle\langle i,k\rangle\langle j\rangle\\ &= q^{-1}\langle j\rangle\langle j+2,k\rangle\langle i,k\rangle-\langle j+2,k\rangle\langle j\rangle\langle i,k\rangle=q^{-1}\langle j,k\rangle\langle i,k\rangle.\end{aligned}$$

Similarly we can prove (5).

Let us prove (6). We have

$$\begin{aligned}\langle i,k\rangle\langle j,\ell\rangle &= \big(\langle i,j-2\rangle\langle j,k\rangle-q\langle j,k\rangle\langle i,j-2\rangle\big)\langle j,\ell\rangle\\ &= q^{-1}\langle i,j-2\rangle\langle j,\ell\rangle\langle j,k\rangle-q\langle j,k\rangle\big(\langle i,\ell\rangle+q\langle j,\ell\rangle\langle i,j-2\rangle\big)\\ &= q^{-1}\big(\langle i,\ell\rangle+q\langle j,\ell\rangle\langle i,j-2\rangle\big)\langle j,k\rangle-q\langle i,\ell\rangle\langle j,k\rangle-q\langle j,\ell\rangle\langle j,k\rangle\langle i,j-2\rangle\\ &= \langle j,\ell\rangle\langle i,k\rangle+(q^{-1}-q)\langle i,\ell\rangle\langle j,k\rangle.\end{aligned}$$

Q.E.D.

Lemma 3.16. (i) *For* $1 \leqslant i \leqslant j$, *we have* $\langle -j,-i\rangle\widetilde{\phi} = \langle i,j\rangle\widetilde{\phi}$.
(ii) *For* $1 \leqslant i < j$, *we have* $\langle -j,i\rangle\widetilde{\phi} = q^{-1}\langle -i,j\rangle\widetilde{\phi}$.

Proof. (i) If $i = j$, it is obvious. By the induction on $j - i$, we have

$$\begin{aligned}\langle -j,-i\rangle\widetilde{\phi} &= (\langle -j,-i-2\rangle\langle -i\rangle - q\langle -i\rangle\langle -j,-i-2\rangle)\widetilde{\phi}\\ &= (\langle -j,-i-2\rangle\langle i\rangle - q\langle -i\rangle\langle i+2,j\rangle)\widetilde{\phi}\\ &= (\langle i\rangle\langle -j,-i-2\rangle - q\langle i+2,j\rangle\langle -i\rangle)\widetilde{\phi}\\ &= (\langle i\rangle\langle i+2,j\rangle - q\langle i+2,j\rangle\langle i\rangle)\widetilde{\phi} = \langle i,j\rangle\widetilde{\phi}.\end{aligned}$$

(ii) By (i), we have

$$\begin{aligned}\langle -j,i\rangle\widetilde{\phi} &= (\langle -j,-1\rangle\langle 1,i\rangle - q\langle 1,i\rangle\langle -j,-1\rangle)\widetilde{\phi}\\ &= (\langle -j,-1\rangle\langle -i,-1\rangle - q\langle 1,i\rangle\langle 1,j\rangle)\widetilde{\phi}\\ &= (q^{-1}\langle -i,-1\rangle\langle -j,-1\rangle - \langle 1,j\rangle\langle 1,i\rangle)\widetilde{\phi}\\ &= (q^{-1}\langle -i,-1\rangle\langle 1,j\rangle - \langle 1,j\rangle\langle -i,-1\rangle)\widetilde{\phi} = q^{-1}\langle -i,j\rangle\widetilde{\phi}.\end{aligned}$$

Q.E.D.

Proposition 3.17. (i) *For a multisegment* $\mathfrak{m} = \sum_{i\leqslant j} m_{i,j}\langle i,j\rangle$, *we have*

$$\mathrm{Ad}(t_k)P(\mathfrak{m}) = q^{\sum_i(m_{i,k-2}-m_{i,k})+\sum_j(m_{k+2,j}-m_{k,j})}P(\mathfrak{m}).$$

(ii)

$$\begin{aligned}e'_k\langle i,j\rangle^{(n)} &= \begin{cases} q^{1-n}\langle i\rangle^{(n-1)} & \text{if } k=i=j,\\ (1-q^2)q^{1-n}\langle i+2,j\rangle\langle i,j\rangle^{(n-1)} & \text{if } k=i<j,\\ 0 & \text{otherwise,}\end{cases}\\ e^*_k\langle i,j\rangle^{(n)} &= \begin{cases} q^{1-n}\langle i\rangle^{(n-1)} & \text{if } i=j=k,\\ (1-q^2)q^{1-n}\langle i,j\rangle^{(n-1)}\langle i,j-2\rangle & \text{if } i<j=k,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Proof. (i) is obvious. Let us show (ii). It is obvious that $e'_k\langle i,j\rangle^{(n)} = 0$ unless $i \leqslant k \leqslant j$. It is known ([K]) that we have $e'_k\langle k\rangle^{(n)} = q^{1-n}\langle k\rangle^{(n-1)}$. We shall prove $e'_k\langle k,j\rangle^{(n)} = (1-q^2)q^{1-n}\langle k+2,j\rangle\langle k,j\rangle^{(n-1)}$ for $k < j$ by the induction on n. By (2.1), we have

$$\begin{aligned}e'_k\langle k,j\rangle &= e'_k(\langle k\rangle\langle k+2,j\rangle - q\langle k+2,j\rangle\langle k\rangle)\\ &= \langle k+2,j\rangle - q^2\langle k+2,j\rangle = (1-q^2)\langle k+2,j\rangle.\end{aligned}$$

For $n \geqslant 1$, by the induction hypothesis and Proposition 3.14 (4), we get

$$\begin{aligned}&[n]e'_k\langle k,j\rangle^{(n)} = e'_k\langle k,j\rangle\langle k,j\rangle^{(n-1)}\\ &\quad= (1-q^2)\langle k+2,j\rangle\langle k,j\rangle^{(n-1)} + q^{-1}\langle k,j\rangle\cdot(1-q^2)q^{2-n}\langle k+2,j\rangle\langle k,j\rangle^{(n-2)}\\ &\quad= (1-q^2)\left\{\langle k+2,j\rangle\langle k,j\rangle^{(n-1)} + q^{1-n}\langle k,j\rangle\langle k+2,j\rangle\langle k,j\rangle^{(n-2)}\right\}\\ &\quad= (1-q^2)(1+q^{-n}[n-1])\langle k+2,j\rangle\langle k,j\rangle^{(n-1)}\\ &\quad= (1-q^2)q^{1-n}[n]\langle k+2,j\rangle\langle k,j\rangle^{(n-1)}.\end{aligned}$$

Finally we show $e'_k\langle i,j\rangle = 0$ if $k \neq i$. We may assume $i < k \leqslant j$. If $i < k < j$, we have

$$\begin{aligned} e'_k\langle i,j\rangle &= e'_k(\langle i,k-2\rangle\langle k,j\rangle - q\langle k,j\rangle\langle i,k-2\rangle) \\ &= q\langle i,k-2\rangle e'_k\langle k,j\rangle - q(e'_k\langle k,j\rangle)\langle i,k-2\rangle \\ &= q(1-q^2)\langle i,k-2\rangle\langle k+2,j\rangle - q(1-q^2)\langle k+2,j\rangle\langle i,k-2\rangle \\ &= 0. \end{aligned}$$

The case $k = j$ is similarly proved.

The proof for e^*_k is similar. Q.E.D.

3.4. **Actions of divided powers.**

Lemma 3.18. *Let a, b be non-negative integers, and let $k \in I_{>0} := \{k \in I \mid k > 0\}$.*

(1) *For $\ell > k$, we have*

$$\begin{aligned} \langle -k\rangle\langle -k+2,\ell\rangle^{(a)}\langle -k,\ell\rangle^{(b)} &= [b+1]\langle -k+2,\ell\rangle^{(a-1)}\langle -k,\ell\rangle^{(b+1)} \\ &\quad + q^{a-b}\langle -k+2,\ell\rangle^{(a)}\langle -k,\ell\rangle^{(b)}\langle -k\rangle. \end{aligned}$$

(2) *We have*

$$\begin{aligned} \langle -k\rangle\langle -k+2,k\rangle^{(a)}\langle -k,k\rangle^{[b]} &= [2b+2]\langle -k+2,k\rangle^{(a-1)}\langle -k,k\rangle^{[b+1]} \\ &\quad + q^{a-b}\langle -k+2,k\rangle^{(a)}\langle -k,k\rangle^{[b]}\langle -k\rangle. \end{aligned}$$

(3) *For $k > 1$, we have*

$$\begin{aligned} \langle -k\rangle\langle -k+2,k-2\rangle^{[a]} &= (q^a+q^{-a})^{-1}\langle -k+2,k-2\rangle^{[a-1]}\langle -k,k-2\rangle \\ &\quad + q^a\langle -k+2,k-2\rangle^{[a]}\langle -k\rangle. \end{aligned}$$

(4) *If $\ell \leqslant k-2$, we have*

$$\langle \ell,k-2\rangle^{(a)}\langle k\rangle = \langle \ell,k\rangle\langle \ell,k-2\rangle^{(a-1)} + q^a\langle k\rangle\langle \ell,k-2\rangle^{(a)}.$$

(5) *For $k > 1$, we have*

$$\begin{aligned} \langle -k+2,k-2\rangle^{[a]}\langle k\rangle &= (q^a+q^{-a})^{-1}\langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[a-1]} \\ &\quad + q^a\langle k\rangle\langle -k+2,k-2\rangle^{[a]}. \end{aligned}$$

Proof. We show (1) by the induction on a. If $a = 0$, it is trivial. For $a > 0$, we have

$$\begin{aligned} &[a]\langle -k\rangle\langle -k+2,\ell\rangle^{(a)}\langle -k,\ell\rangle^{(b)} \\ =\ &\big(\langle -k,\ell\rangle + q\langle -k+2,\ell\rangle\langle -k\rangle\big)\langle -k+2,\ell\rangle^{(a-1)}\langle -k,\ell\rangle^{(b)} \\ =\ &[b+1]q^{1-a}\langle -k+2,\ell\rangle^{(a-1)}\langle -k,\ell\rangle^{(b+1)} \\ &+q\langle -k+2,\ell\rangle\{[b+1]\langle -k+2,\ell\rangle^{(a-2)}\langle -k,\ell\rangle^{(b+1)} + q^{a-b-1}\langle -k+2,\ell\rangle^{(a-1)}\langle -k,\ell\rangle^{(b)}\langle -k\rangle\} \\ =\ &[b+1](q^{1-a}+q[a-1])\langle -k+2,\ell\rangle^{(a-1)}\langle -k,\ell\rangle^{(b+1)} + q^{a-b}[a]\langle -k+2,\ell\rangle^{(a)}\langle -k,\ell\rangle^{(b)}\langle -k\rangle. \end{aligned}$$

Since $q^{1-a} + q[a-1] = [a]$, the induction proceeds.

The proof of (2) is similar by using $\langle -k,k\rangle^{[b]} = [2b]\langle -k,k\rangle^{[b-1]}\langle -k,k\rangle$.

We prove (3) by the induction on a. The case $a=0$ is trivial. For $a>0$, we have

$$
\begin{aligned}
&[2a]\langle -k\rangle\langle -k+2,k-2\rangle^{[a]} = \big(\langle -k,k-2\rangle + q\langle -k+2,k-2\rangle\langle -k\rangle\big)\langle -k+2,k-2\rangle^{[a-1]}\\
&\quad = q^{1-a}\langle -k+2,k-2\rangle^{[a-1]}\langle -k,k-2\rangle\\
&\qquad + q\langle -k+2,k-2\rangle\big\{(q^{a-1}+q^{1-a})^{-1}\langle -k+2,k-2\rangle^{[a-2]}\langle -k,k-2\rangle\\
&\qquad\qquad\qquad\qquad + q^{a-1}\langle -k+2,k-2\rangle^{[a-1]}\langle -k\rangle\big\}\\
&= \Big(q^{1-a}+\frac{q[2a-2]}{q^{a-1}+q^{1-a}}\Big)\langle -k+2,k-2\rangle^{[a-1]}\langle -k,k-2\rangle + q^{a}[2a]\langle -k+2,k-2\rangle^{[a]}\langle -k\rangle\\
&= (q^{a}+q^{-a})^{-1}[2a]\langle -k+2,k-2\rangle^{[a-1]}\langle -k,k-2\rangle + q^{a}[2a]\langle -k+2,k-2\rangle^{[a]}\langle -k\rangle.
\end{aligned}
$$

Similarly, we can prove (4) and (5) by the induction on a. Q.E.D.

Lemma 3.19. *For $k>1$ and $a,b,c,d\geqslant 0$, set*

$$(a,b,c,d)=\langle k\rangle^{(a)}\langle -k+2,k\rangle^{(b)}\langle -k,k\rangle^{[c]}\langle -k+2,k-2\rangle^{[d]}\widetilde{\phi}.$$

Then, we have

$$
\begin{aligned}
\langle -k\rangle(a,b,c,d) &= [2c+2](a,b-1,c+1,d)\\
&\quad +[b+1]q^{b-2c}(a,b+1,c,d-1)\\
&\quad +[a+1]q^{2d-2c}(a+1,b,c,d).
\end{aligned}
\tag{3.4}
$$

Proof. We shall show first

$$
\begin{aligned}
&\langle -k\rangle\langle -k+2,k-2\rangle^{[d]}\widetilde{\phi}\\
&\qquad = \big(\langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[d-1]} + q^{2d}\langle k\rangle\langle -k+2,k-2\rangle^{[d]}\big)\widetilde{\phi}.
\end{aligned}
\tag{3.5}
$$

By Lemma 3.18 (3), we have

$$
\begin{aligned}
\langle -k\rangle\langle -k+2,k-2\rangle^{[d]}\widetilde{\phi} &= \big((q^{d}+q^{-d})^{-1}\langle -k+2,k-2\rangle^{[d-1]}\langle -k,k-2\rangle\\
&\qquad + q^{d}\langle -k+2,k-2\rangle^{[d]}\langle -k\rangle\big)\widetilde{\phi}.
\end{aligned}
$$

By Lemma 3.16 and Lemma 3.18 (5), it is equal to

$$
\begin{aligned}
&\big((q^{d}+q^{-d})^{-1}q^{-1}\langle -k+2,k-2\rangle^{[d-1]}\langle -k+2,k\rangle + q^{d}\langle -k+2,k-2\rangle^{[d]}\langle k\rangle\big)\widetilde{\phi}\\
&= \Big((q^{d}+q^{-d})^{-1}q^{-1}q^{1-d}\langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[d-1]}\\
&\quad + q^{d}\big((q^{d}+q^{-d})^{-1}\langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[d-1]} + q^{d}\langle k\rangle\langle -k+2,k-2\rangle^{[d]}\big)\Big)\widetilde{\phi}.
\end{aligned}
$$

Thus we obtain (3.5). Applying Lemma 3.18 (2), we have

$$
\begin{aligned}
\langle -k\rangle(a,b,c,d) &= \langle k\rangle^{(a)}\Big([2c+2]\langle -k+2,k\rangle^{(b-1)}\langle -k,k\rangle^{[c+1]}\\
&\qquad + q^{b-c}\langle -k+2,k\rangle^{(b)}\langle -k,k\rangle^{[c]}\langle -k\rangle\Big)\langle -k+2,k-2\rangle^{[d]}\widetilde{\phi}\\
&= [2c+2](a,b-1,c+1,d) + q^{b-c}\langle k\rangle^{(a)}\langle -k+2,k\rangle^{(b)}\langle -k,k\rangle^{[c]}\\
&\qquad \times\big(\langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[d-1]} + q^{2d}\langle k\rangle\langle -k+2,k-2\rangle^{[d]}\big)\widetilde{\phi}\\
&= [2c+2](a,b-1,c+1,d) + q^{b-2c}[b+1](a,b+1,c,d-1)\\
&\qquad + q^{(b-c)+2d-c-b}[a+1](a+1,b,c,d).
\end{aligned}
$$

Hence we have (3.4). Q.E.D.

Proposition 3.20. (1) *We have*

$$\langle -1\rangle^{(a)}\langle -1,1\rangle^{[m]}\widetilde{\phi}=\sum_{s=0}^{\lfloor a/2\rfloor}\left(\prod_{\nu=1}^{s}\frac{[2m+2\nu]}{[2\nu]}\right)q^{-2(a-s)m+\frac{(a-2s)(a-2s-1)}{2}}\langle 1\rangle^{(a-2s)}\langle -1,1\rangle^{[m+s]}\widetilde{\phi}.$$

(2) *For $k>1$, we have*

$$\begin{aligned}&\langle -k\rangle^{(n)}\langle -k+2,k-2\rangle^{[a]}\widetilde{\phi}\\ =&\sum_{u=0}^{n}\sum_{i+j+2t=n,j+t=u}q^{2ai+\frac{j(j-1)}{2}-i(t+u)}\langle k\rangle^{(i)}\langle -k+2,k\rangle^{(j)}\langle -k,k\rangle^{[t]}\langle -k+2,k-2\rangle^{[a-u]}\widetilde{\phi}.\end{aligned}$$

(3) *If $\ell>k$, we have*

$$\langle k\rangle^{(n)}\langle k+2,\ell\rangle^{(a)}=\sum_{s=0}^{n}q^{(n-s)(a-s)}\langle k+2,\ell\rangle^{(a-s)}\langle k,\ell\rangle^{(s)}\langle k\rangle^{(n-s)}.$$

Proof. We prove (1) by the induction on a. The case $a=0$ is trivial. Assume $a>0$. Then, Lemma 3.18 (2) implies

$$\begin{aligned}\langle -1\rangle\langle 1\rangle^{(n)}\langle -1,1\rangle^{[m]}\widetilde{\phi}&=\big([2m+2]\langle 1\rangle^{(n-1)}\langle -1,1\rangle^{[m+1]}+q^{n-m}\langle 1\rangle^{(n)}\langle -1,1\rangle^{[m]}\langle -1\rangle\big)\widetilde{\phi}\\&=\big([2m+2]\langle 1\rangle^{(n-1)}\langle -1,1\rangle^{[m+1]}+q^{n-m}\langle 1\rangle^{(n)}\langle -1,1\rangle^{[m]}\langle 1\rangle\big)\widetilde{\phi}\\&=\big([2m+2]\langle 1\rangle^{(n-1)}\langle -1,1\rangle^{[m+1]}+q^{n-2m}[n+1]\langle 1\rangle^{(n+1)}\langle -1,1\rangle^{[m]}\big)\widetilde{\phi}.\end{aligned}$$

Put

$$c_s=\left(\prod_{\nu=1}^{s}\frac{[2m+2\nu]}{[2\nu]}\right)q^{-2(a-s)m+\frac{(a-2s)(a-2s-1)}{2}}.$$

Then we have

$$\begin{aligned}[a+1]\langle -1\rangle^{(a+1)}\langle -1,1\rangle^{[m]}\widetilde{\phi}&=\langle -1\rangle\langle -1\rangle^{(a)}\langle -1,1\rangle^{[m]}\widetilde{\phi}\\&=\langle -1\rangle\sum_{s=0}^{\lfloor a/2\rfloor}c_s\langle 1\rangle^{(a-2s)}\langle -1,1\rangle^{[m+s]}\widetilde{\phi}\\&=\sum_{s=0}^{\lfloor a/2\rfloor}c_s\big\{[2(m+s+1)]\langle 1\rangle^{(a-2s-1)}\langle -1,1\rangle^{[m+s+1]}\\&\qquad+q^{a-2s-2(m+s)}[a-2s+1]\langle 1\rangle^{(a-2s+1)}\langle -1,1\rangle^{[m+s]}\big\}\widetilde{\phi}.\end{aligned}$$

In the right-hand-side, the coefficients of $\langle 1\rangle^{a+1-2r}\langle -1,1\rangle^{[m+r]}\widetilde{\phi}$ are

$$\begin{aligned}&[2(m+r)]c_{r-1}+q^{a-2m-4r}[a-2r+1]c_r\\&=\prod_{\nu=1}^{r}\frac{[2m+2\nu]}{[2\nu]}q^{-2(a-r+1)m+\frac{(a-2r)(a-2r+1)}{2}}\Big([2r]q^{a-2r+1}+[a-2r+1]q^{-2r}\Big)\\&=[a+1]\prod_{\nu=1}^{r}\frac{[2m+2\nu]}{[2\nu]}q^{-2(a-r+1)m+\frac{(a-2r)((a-2r+1)}{2}}.\end{aligned}$$

Hence we obtain (1).

We prove (2) by the induction on n. We use the following notation for short:

$$(i,j,t,a):=\langle k\rangle^{(i)}\langle -k+2,k\rangle^{(j)}\langle -k,k\rangle^{[t]}\langle -k+2,k-2\rangle^{[a]}\widetilde{\phi}.$$

Then Lemma 3.19 implies that

$$\begin{aligned}\langle -k\rangle(i,j,t,a) &= [2t+2](i,j-1,t+1,a)\\ &\quad+[j+1]q^{j-2t}(i,j+1,t,a-1)\\ &\quad+[i+1]q^{2a-2t}(i+1,j,t,a).\end{aligned}$$

Hence, by assuming (2) for n, we have

$$[n+1]\langle -k\rangle^{(n+1)}\langle -k+2,k-2\rangle^{[a]}\widetilde{\phi} = \langle -k\rangle\langle -k\rangle^{(n)}\langle -k+2,k-2\rangle^{[a]}\widetilde{\phi}$$
$$= \sum_{u=0}^{n}\sum_{i+j+2t=n,j+t=u}\left\{\begin{array}{l}[2t+2]q^{2ai+\frac{j(j-1)}{2}-i(t+u)}(i,j-1,t+1,a-u)\\ +[j+1]q^{2ai+\frac{j(j-1)}{2}-i(t+u)+j-2t}(i,j+1,t,a-u-1)\\ +[i+1]q^{2ai+\frac{j(j-1)}{2}-i(t+u)+2a-2u-2t}(i+1,j,t,a-u)\end{array}\right\}.$$

Then in the right hand side, the coefficients of $(i',j',t',a-u')$ satisfying $i'+j'+2t'=n+1, j'+t'=u'$ are

$$\begin{aligned}&[2t']q^{2ai'+\frac{(j'+1)j'}{2}-i'(t'-1+u')}+[j']q^{2ai'+\frac{(j'-1)(j'-2)}{2}-i'(t'+u'-1)+j'-1-2t'}\\ &\qquad\qquad+[i']q^{2a(i'-1)+\frac{j'(j'-1)}{2}-(i'-1)(t'+u')+2a-2u'-2t'}\\ &= q^{2ai'+\frac{j'(j'-1)}{2}-i'(t'+u')}\Big([2t']q^{j'+i'}+[j']q^{i'-2t'}+[i']q^{-(t'+u')}\Big)\\ &= q^{2ai'+\frac{j'(j'-1)}{2}-i'(t'+u')}[n+1].\end{aligned}$$

We can prove (3) similarly as above. Q.E.D.

3.5. **Actions of E_k, F_k on the PBW basis.** For a θ-restricted multisegment $\mathfrak{m}$, we set

$$\widetilde{P}_\theta(\mathfrak{m}) = P_\theta(\mathfrak{m})\widetilde{\phi}.$$

We understand $\widetilde{P}_\theta(\mathfrak{m})=0$ if $\mathfrak{m}$ is not a multisegment.

Theorem 3.21. *For $k\in I_{>0}$ and a θ-restricted multisegment $\mathfrak{m}=\sum_{-j\leqslant i\leqslant j}m_{i,j}\langle i,j\rangle$, we have*

$$\begin{aligned}&F_{-k}\widetilde{P}_\theta(\mathfrak{m})\\ &=\sum_{\ell>k}[m_{-k,\ell}+1]q^{\sum\limits_{\ell'>\ell}(m_{-k+2,\ell'}-m_{-k,\ell'})}\widetilde{P}_\theta(\mathfrak{m}-\langle -k+2,\ell\rangle+\langle -k,\ell\rangle)\\ &\quad+q^{\sum\limits_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})}[2m_{-k,k}+2]\widetilde{P}_\theta(\mathfrak{m}-\langle -k+2,k\rangle+\langle -k,k\rangle)\\ &\quad+q^{\sum\limits_{\ell>k}(m_{-k+2,k}-m_{-k,k})+m_{-k+2,k}-2m_{-k,k}}[m_{-k+2,k}+1]\widetilde{P}_\theta(\mathfrak{m}-\delta_{k\neq1}\langle -k+2,k-2\rangle+\langle -k+2,k\rangle)\\ &\quad+\sum_{-k+2<i\leqslant k}q^{\sum\limits_{\ell>k}(m_{-k+2,k}-m_{-k,k})+2m_{-k+2,k-2}-2m_{-k,k}+\sum\limits_{-k+2<j<i}(m_{j,k-2}-m_{j,k})}\\ &\qquad\qquad\qquad\times[m_{i,k}+1]\widetilde{P}_\theta(\mathfrak{m}-\delta_{i<k}\langle i,k-2\rangle+\langle i,k\rangle).\end{aligned}$$

Proof. We divide $\mathfrak{m}$ into four parts, $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3+\delta_{k\neq1}m_{-k+2,k-2}\langle -k+2,k-2\rangle$, where $\mathfrak{m}_1=\sum_{j>k}m_{i,j}\langle i,j\rangle$, $\mathfrak{m}_2=\sum_{j=k}m_{i,j}\langle i,j\rangle$, $\mathfrak{m}_3=\sum_{-k+2<i\leqslant j\leqslant k-2}m_{i,j}\langle i,j\rangle$. Then Proposition 3.14 implies

$$\widetilde{P}_\theta(\mathfrak{m}) = P_\theta(\mathfrak{m}_1)P_\theta(\mathfrak{m}_2)P_\theta(\mathfrak{m}_3)\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi}.$$

If $k=1$, we understand $\langle -k+2, k-2\rangle^{[n]}=1$. By Lemma 3.18 (1), we have

$$\langle -k\rangle P_\theta(\mathfrak{m}_1)=\sum_{\ell>k} q^{\sum_{\ell'>\ell}(m_{-k+2,\ell'}-m_{-k,\ell'})}[m_{-k,\ell}+1]P_\theta(\mathfrak{m}_1-\langle -k+2,\ell\rangle+\langle -k,\ell\rangle)$$
$$+q^{\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})}P_\theta(\mathfrak{m}_1)\langle -k\rangle,$$

and Lemma 3.18 (2) implies

$$\langle -k\rangle P_\theta(\mathfrak{m}_2)=[2m_{-k,k}+2]P_\theta(\mathfrak{m}_2-\langle -k+2,k\rangle+\langle -k,k\rangle)$$
$$+q^{m_{-k+2,k}-m_{-k,k}}P_\theta(\mathfrak{m}_2)\langle -k\rangle.$$

Since we have $\langle -k\rangle P_\theta(\mathfrak{m}_3)=P_\theta(\mathfrak{m}_3)\langle -k\rangle$, we obtain

$$\begin{aligned}\langle -k\rangle \widetilde{P}_\theta(\mathfrak{m})= \;&\textstyle\sum_{\ell>k} q^{\sum_{\ell'>\ell}(m_{-k+2,\ell'}-m_{-k,\ell'})}[m_{-k,\ell}+1]\widetilde{P}_\theta(\mathfrak{m}-\langle -k+2,\ell\rangle+\langle -k,\ell\rangle)\\ &+q^{\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})}[2m_{-k,k}+2]\widetilde{P}_\theta(\mathfrak{m}-\langle -k+2,k\rangle+\langle -k,k\rangle)\\ &+q^{\sum_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})}P_\theta(\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3)\langle -k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi}.\end{aligned}\tag{3.6}$$

By (3.5), we have

$$\begin{aligned}\langle -k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi} \;=\;& \langle -k+2,k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}-1]}\widetilde{\phi}\\ &+\delta_{k\neq 1}q^{2m_{-k+2,k-2}}\langle k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi}.\end{aligned}$$

Hence the last term in (3.6) is equal to

$$q^{\sum_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})-m_{-k,k}}[m_{-k+2,k}+1]\widetilde{P}_\theta(\mathfrak{m}-\delta_{k\neq 1}\langle -k+2,k-2\rangle+\langle -k+2,k\rangle)$$
$$+\delta_{k\neq 1}q^{\sum_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})+2m_{-k+2,k-2}}P_\theta(\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3)\langle k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi}.$$

For $k\neq 1$, Lemma 3.18 (4) implies

$$P_\theta(\mathfrak{m}_3)\langle k\rangle=\sum_{-k+2<i\leqslant k} q^{\sum_{-k+2<j<i}m_{j,k-2}}\langle i,k\rangle P_\theta(\mathfrak{m}_3-\delta_{i<k}\langle i,k-2\rangle),$$

and Proposition 3.14 implies

$$P_\theta(\mathfrak{m}_2)\langle i,k\rangle=q^{-\sum_{j<i}m_{j,k}}[m_{i,k}+1]P_\theta(\mathfrak{m}_2+\langle i,k\rangle).$$

Hence we obtain

$$\begin{aligned}&P_\theta(\mathfrak{m}_1)P_\theta(\mathfrak{m}_2)P_\theta(\mathfrak{m}_3)\langle k\rangle\langle -k+2,k-2\rangle^{[m_{-k+2,k-2}]}\widetilde{\phi}\\ &\quad=\sum_{-k+2<i\leqslant k} q^{\sum_{-k+2<j<i}m_{j,k-2}-\sum_{-k\leqslant j<i}m_{j,k}}[m_{i,k}+1]\widetilde{P}_\theta(\mathfrak{m}-\delta_{i<k}\langle i,k-2\rangle+\langle i,k\rangle).\end{aligned}$$

Thus we obtain the desired result. Q.E.D.

Theorem 3.22. *For $k \in I_{>0}$ and a θ-restricted multisegment $\mathfrak{m} = \sum_{-j \leqslant i \leqslant j} m_{i,j}\langle i, j\rangle$, we have*

$$
\begin{aligned}
& E_{-k}\widetilde{P}_\theta(\mathfrak{m}) \\
&= (1-q^2)\sum_{\ell>k} q^{1+\sum_{\ell'\geqslant \ell}(m_{-k+2,\ell'}-m_{-k,\ell'})}[m_{-k+2,\ell}+1]\widetilde{P}_\theta(\mathfrak{m}-\langle -k,\ell\rangle+\langle -k+2,\ell\rangle) \\
&+(1-q^2)q^{1+\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})+m_{-k+2,k}-2m_{-k,k}}[m_{-k+2,k}+1]\widetilde{P}_\theta(\mathfrak{m}-\langle -k,k\rangle+\langle -k+2,k\rangle) \\
&+(1-q^2)\sum_{-k+2<i\leqslant k-2} q^{1+\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})+2m_{-k+2,k-2}-2m_{-k,k}+\sum_{-k+2<i'\leqslant i}(m_{i,k-2}-m_{i'k})} \\
&\qquad\qquad \times[m_{i,k-2}+1]\widetilde{P}_\theta(\mathfrak{m}-\langle i,k\rangle+\langle i,k-2\rangle) \\
&+\delta_{k\neq 1}(1-q^2)q^{1+\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})+2m_{-k+2,k-2}-2m_{-k,k}} \\
&\qquad\qquad \times[2(m_{-k+2,k-2}+1)]\widetilde{P}_\theta(\mathfrak{m}-\langle -k+2,k\rangle+\langle -k+2,k-2\rangle) \\
&+q^{\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})-2m_{-k,k}+\delta_{k\neq1}\left(1-m_{k,k}+2m_{-k+2,k-2}+\sum_{-k+2<i\leqslant k-2}(m_{i,k-2}-m_{i,k})\right)}\widetilde{P}_\theta(\mathfrak{m}-\langle k\rangle).
\end{aligned}
$$

Proof. We shall divide $\mathfrak{m}$ into $\mathfrak{m}=\mathfrak{m}_1+\mathfrak{m}_2+\mathfrak{m}_3$ where $\mathfrak{m}_1 = \sum_{i\leqslant j, j>k} m_{i,j}\langle i,j\rangle$ and $\mathfrak{m}_2 = \sum_{i\leqslant k} m_{i,k}\langle i,k\rangle$ and $\mathfrak{m}_3 = \sum_{i\leqslant j<k} m_{i,j}\langle i,j\rangle$. By (2.3) and Proposition 3.17, we have

$$
\begin{aligned}
E_{-k}\widetilde{P}_\theta(\mathfrak{m}) &= \Big(\big(e'_{-k}P_\theta(\mathfrak{m}_1)\big)P_\theta(\mathfrak{m}_2+\mathfrak{m}_3)+(\mathrm{Ad}(t_{-k})P_\theta(\mathfrak{m}_1))(e'_{-k}P_\theta(\mathfrak{m}_2+\mathfrak{m}_3)) \\
&\quad + \mathrm{Ad}(t_{-k})\left\{P_\theta(\mathfrak{m}_1)\big(e^*_kP_\theta(\mathfrak{m}_2)\big)\,\mathrm{Ad}(t_k)P_\theta(\mathfrak{m}_3)\right\}\Big)\widetilde{\phi}.
\end{aligned} \tag{3.7}
$$

By Proposition 3.17, the first term is

$$
\begin{aligned}
&\big(e'_{-k}P_\theta(\mathfrak{m}_1)\big)P_\theta(\mathfrak{m}_2+\mathfrak{m}_3) \\
&\quad = (1-q^2)\textstyle\sum_{\ell>k} q^{1+\sum_{\ell'\geqslant\ell}(m_{-k+2,\ell'}-m_{-k,\ell'})} \\
&\qquad\qquad \times[m_{-k+2,\ell}+1]P_\theta(\mathfrak{m}-\langle -k,\ell\rangle+\langle -k+2,\ell\rangle).
\end{aligned} \tag{3.8}
$$

The second term is

$$
\begin{aligned}
&(\mathrm{Ad}(t_{-k})P_\theta(\mathfrak{m}_1))(e'_{-k}P_\theta(\mathfrak{m}_2+\mathfrak{m}_3)) \\
&= q^{\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})}\frac{[m_{-k,k}][m_{-k+2,k}+1]}{[2m_{-k,k}]}(1-q^2)q^{1-m_{-k,k}+m_{-k+2,k}} \\
&\qquad \times P_\theta(\mathfrak{m}-\langle -k,k\rangle+\langle -k+2,k\rangle).
\end{aligned} \tag{3.9}
$$

Let us calculate the last part of (3.7). We have

$$
\begin{aligned}
&\mathrm{Ad}(t_{-k})\Big(P_\theta(\mathfrak{m}_1)\big(e^*_kP_\theta(\mathfrak{m}_2)\big)\,\mathrm{Ad}(t_k)P_\theta(\mathfrak{m}_3)\Big) \\
&\quad = q^{\sum_\ell(m_{-k+2,\ell}-m_{-k,\ell})+\sum_{i\leqslant k-2}m_{i,k-2}-\delta_{k=1}}P_\theta(\mathfrak{m}_1)\big(e^*_kP_\theta(\mathfrak{m}_2)\big)P_\theta(\mathfrak{m}_3).
\end{aligned}
$$

We have

$$
\begin{aligned}
e^*_kP_\theta(\mathfrak{m}_2) &= q^{1-m_k-\sum_{i<k}m_{i,k}}P_\theta(\mathfrak{m}_2-\langle k\rangle) \\
&\quad +(1-q^2)\sum_{-k<i<k} q^{1-m_{i,k}-\sum_{i'<i}m_{i',k}}P_\theta(\mathfrak{m}_2-\langle i,k\rangle)\langle i,k-2\rangle \\
&\quad +\frac{[m_{-k,k}]}{[2m_{-k,k}]}(1-q^2)q^{1-m_{-k,k}}P(\mathfrak{m}_2-\langle -k,k\rangle)\langle -k,k-2\rangle.
\end{aligned}
$$

For $-k<i<k$, we have

$$\langle i,k-2\rangle P_\theta(\mathfrak{m}_3)=q^{-\sum\limits_{i'>i}m_{i',k-2}}[(1+\delta_{i=-k+2})(m_{i,k-2}+1)]P_\theta(\mathfrak{m}_3+\langle i,k-2\rangle).$$

By Lemma 3.16, we have

$$\begin{aligned}
\langle -k,k-2\rangle P_\theta(\mathfrak{m}_3)\widetilde{\phi} &= q^{-\sum\limits_{-k+2\leqslant k\leqslant k-2}m_{i,k-2}}P_\theta(\mathfrak{m}_3)\langle -k,k-2\rangle\widetilde{\phi}\\
&= q^{-\sum\limits_{-k+2\leqslant k\leqslant k-2}m_{i,k-2}-\delta_{k\neq 1}}P_\theta(\mathfrak{m}_3)\langle -k+2,k\rangle\widetilde{\phi}\\
&= q^{-m_{-k+2,k-2}-\sum\limits_{-k+2\leqslant i\leqslant k-2}m_{i,k-2}-\delta_{k\neq 1}}\langle -k+2,k\rangle P_\theta(\mathfrak{m}_3)\widetilde{\phi}.
\end{aligned}$$

Hence we obtain

$$\begin{aligned}
&P_\theta(\mathfrak{m}_1)\big(e_k^* P_\theta(\mathfrak{m}_2)\big)P_\theta(\mathfrak{m}_3)\widetilde{\phi}\\
&=q^{1-\sum\limits_{i\leqslant k}m_{i,k}}\widetilde{P}_\theta(\mathfrak{m}-\langle k\rangle)\\
&+(1-q^2)\sum_{-k+2<i\leqslant k-2}q^{1-\sum\limits_{i'\leqslant i}m_{i',k}-\sum\limits_{i'>i}m_{i',k-2}}\\
&\qquad\times[m_{i,k-2}+1]\widetilde{P}_\theta(\mathfrak{m}-\langle i,k\rangle+\langle i,k-2\rangle)\\
&+(1-q^2)\delta_{k\neq 1}q^{1-m_{-k,k}-m_{-k+2,k}-\sum\limits_{-k+2<i}m_{i,k-2}}\\
&\qquad\times[2(m_{-k+2,k-2}+1)]\widetilde{P}_\theta(\mathfrak{m}-\langle -k+2,k\rangle+\langle -k+2,k-2\rangle)\\
&+(1-q^2)q^{2(1-m_{-k,k})-m_{-k+2,k-2}-\sum\limits_{-k+2\leqslant i\leqslant k-2}m_{i,k-2}-\delta_{k\neq 1}}\\
&\qquad\times\frac{[m_{-k+2,k}+1][m_{-k,k}]}{[2m_{-k,k}]}P(\mathfrak{m}-\langle -k,k\rangle+\langle -k+2,k\rangle).
\end{aligned}$$

Hence the coefficient of $\widetilde{P}_\theta(\mathfrak{m}-\langle k\rangle)$ in $E_{-k}\widetilde{P}_\theta(\mathfrak{m})$ is

$$\begin{aligned}
&q^{\sum\limits_{\ell}(m_{-k+2,\ell}-m_{-k,\ell})+\sum\limits_{i\leqslant k-2}m_{i,k-2}-\delta_{k=1}+1-\sum\limits_{i\leqslant k}m_{i,k}}\\
&=q^{\sum\limits_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})-2m_{-k,k}+\delta_{k\neq 1}\big(1-m_{k,k}+2m_{-k+2,k-2}+\sum\limits_{-k+2<i\leqslant k-2}(m_{i,k-2}-m_{i,k})\big)}.
\end{aligned}$$

The coefficient of $\widetilde{P}_\theta(\mathfrak{m}-\langle -k,k\rangle+\langle -k+2,k\rangle)$ in $E_{-k}\widetilde{P}_\theta(\mathfrak{m})$ is

$$\begin{aligned}
&(1-q^2)q^{1+\sum\limits_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})}\frac{[m_{-k,k}][m_{-k+2,k}+1]}{[2m_{-k,k}]}\\
&+(1-q^2)q^{\sum\limits_{\ell}(m_{-k+2,\ell}-m_{-k,\ell})+\sum\limits_{i\leqslant k-2}m_{i,k-2}-\delta_{k=1}+2(1-m_{-k,k})-m_{-k+2,k-2}-\sum\limits_{-k+2\leqslant i\leqslant k-2}m_{i,k-2}-\delta_{k\neq 1}}\\
&\qquad\times\frac{[m_{-k+2,k}+1][m_{-k,k}]}{[2m_{-k,k}]}\\
&=(1-q^2)q^{1+\sum_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})}\frac{[m_{-k,k}][m_{-k+2,k}+1]}{[2m_{-k,k}]}(1+q^{-2m_{-k,k}})\\
&=(1-q^2)q^{1-m_{-k,k}+\sum_{\ell\geqslant k}(m_{-k+2,\ell}-m_{-k,\ell})}[m_{-k+2,k}+1]\\
&=(1-q^2)q^{1+m_{-k+2,k}-2m_{-k,k}+\sum_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})}[m_{-k+2,k}+1].
\end{aligned}$$

For $-k+2 < i \leqslant k-2$, the coefficient of $\widetilde{P}_\theta(\mathfrak{m} - \langle i,k\rangle + \langle i,k-2\rangle)$ in $E_{-k}\widetilde{P}_\theta(\mathfrak{m})$ is

$$\begin{aligned}
&(1-q^2)q^{\sum\limits_{\ell}(m_{-k+2,\ell}-m_{-k,\ell})+\sum\limits_{i'\leqslant k-2} m_{i',k-2}-\delta_{k=1}+1-\sum\limits_{i'\leqslant i} m_{i',k}-\sum\limits_{i'>i} m_{i',k-2}}[m_{i,k-2}+1]\\
&= (1-q^2)q^{1+\sum\limits_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})+2m_{-k+2,k-2}-2m_{-k,k}+\sum\limits_{-k+2<i'\leqslant i}(m_{i,k-2}-m_{i'k})}[m_{i,k-2}+1].
\end{aligned}$$

Finally, for $k \neq 1$, the coefficient of $\widetilde{P}_\theta(\mathfrak{m} - \langle -k+2,k\rangle + \langle -k+2,k-2\rangle)$ in $E_{-k}\widetilde{P}_\theta(\mathfrak{m})$ is

$$\begin{aligned}
&(1-q^2)q^{\sum\limits_{\ell}(m_{-k+2,\ell}-m_{-k,\ell})+\sum\limits_{i\leqslant k-2} m_{i,k-2}-\delta_{k=1}+1-m_{-k,k}-m_{-k+2,k}-\sum\limits_{-k+2<i} m_{i,k-2}}[2(m_{-k+2,k-2}+1)]\\
&= (1-q^2)q^{1+\sum\limits_{\ell>k}(m_{-k+2,\ell}-m_{-k,\ell})+2m_{-k+2,k-2}-2m_{-k,k}}[2(m_{-k+2,k-2}+1)].
\end{aligned}$$

Q.E.D.

Theorem 3.23. *For $k>0$ and $\mathfrak{m} \in \mathcal{M}_\theta$, we have*

$$\begin{aligned}
E_k\widetilde{P}_\theta(\mathfrak{m}) &= \textstyle\sum_{\ell>k}(1-q^2)q^{1+\sum_{\ell'\geqslant\ell}(m_{k+2,\ell'}-m_{k,\ell'})}[m_{k+2,\ell}+1]\widetilde{P}_\theta(\mathfrak{m}-\langle k,\ell\rangle+\langle k+2,\ell\rangle)\\
&\qquad +q^{1+\sum_{\ell>k}(m_{k+2,\ell}-m_{k,\ell})-m_{k,k}}\widetilde{P}_\theta(\mathfrak{m}-\langle k\rangle),\\
F_k\widetilde{P}_\theta(\mathfrak{m}) &= \sum_{\ell\geqslant k} q^{\sum_{\ell'>\ell}(m_{k+2,\ell'}-m_{k,\ell'})}[m_{k,\ell}+1]\widetilde{P}_\theta(\mathfrak{m}-\delta_{\ell\neq k}\langle k+2,\ell\rangle+\langle k,\ell\rangle).
\end{aligned}$$

Proof. The first follows from $e^*_{-k}P_\theta(\mathfrak{m})=0$ and Proposition 3.17, and the second follows from Proposition 3.20. Q.E.D.

4. Crystal basis of $V_\theta(0)$

4.1. Crystal structure on $\mathcal{M}_\theta$.

We shall define the crystal structure on $\mathcal{M}_\theta$.

Definition 4.1. *Suppose $k>0$. For a θ-restricted multisegment $\mathfrak{m} = \sum\limits_{-j\leqslant i\leqslant j} m_{i,j}\langle i,j\rangle$, we set*

$$\varepsilon_{-k}(\mathfrak{m}) = \max\left\{A_j^{(-k)}(\mathfrak{m}) \mid j \geqslant -k+2\right\},$$

where

$$\begin{aligned}
A_j^{(-k)}(\mathfrak{m}) &= \sum_{\ell\geqslant j}(m_{-k,\ell}-m_{-k+2,\ell+2}) \quad \textit{for } j>k,\\
A_k^{(-k)}(\mathfrak{m}) &= \sum_{\ell>k}(m_{-k,\ell}-m_{-k+2,\ell})+2m_{-k,k}+\delta(m_{-k+2,k} \textit{ is odd}),\\
A_j^{(-k)}(\mathfrak{m}) &= \sum_{\ell>k}(m_{-k,\ell}-m_{-k+2,\ell})+2m_{-k,k}-2m_{-k+2,k-2}+\sum_{-k+2<i\leqslant j+2} m_{i,k}-\sum_{-k+2<i\leqslant j} m_{i,k-2}\\
&\qquad \textit{for } -k+2\leqslant j\leqslant k-2.
\end{aligned}$$

(i) *Let n_f be the smallest $\ell \geqslant -k+2$, with respect to the ordering $\cdots > k+2 > k > -k+2 > \cdots > k-2$, such that $\varepsilon_{-k}(\mathfrak{m}) = A_\ell^{(-k)}(\mathfrak{m})$. We define*

$$\widetilde{F}_{-k}(\mathfrak{m}) = \begin{cases}
\mathfrak{m}-\langle -k+2,n_f\rangle+\langle -k,n_f\rangle & \textit{if } n_f>k,\\
\mathfrak{m}-\langle -k+2,k\rangle+\langle -k,k\rangle & \textit{if } n_f=k \textit{ and } m_{-k+2,k} \textit{ is odd},\\
\mathfrak{m}-\delta_{k\neq1}\langle -k+2,k-2\rangle+\langle -k+2,k\rangle & \textit{if } n_f=k \textit{ and } m_{-k+2,k} \textit{ is even},\\
\mathfrak{m}-\delta_{n_f\neq k-2}\langle n_f+2,k-2\rangle+\langle n_f+2,k\rangle & \textit{if } -k+2\leqslant n_f\leqslant k-2.
\end{cases}$$

(ii) *If* $\varepsilon_{-k}(\mathfrak{m}) = 0$, *then* $\widetilde{E}_{-k}(\mathfrak{m}) = 0$. *If* $\varepsilon_{-k}(\mathfrak{m}) > 0$, *then let* n_e *be the largest* $\ell \geqslant -k+2$, *with respect to the above ordering, such that* $\varepsilon_{-k}(\mathfrak{m}) = A_\ell^{(-k)}(\mathfrak{m})$. *We define*

$$\widetilde{E}_{-k}(\mathfrak{m}) = \begin{cases} \mathfrak{m} - \langle -k, n_e\rangle + \langle -k+2, n_e\rangle & \text{if } n_e > k, \\ \mathfrak{m} - \langle -k, k\rangle + \langle -k+2, k\rangle & \text{if } n_e = k \text{ and } m_{-k+2,k} \text{ is even,} \\ \mathfrak{m} - \langle -k+2, k\rangle + \delta_{k\neq 1}\langle -k+2, k-2\rangle & \text{if } n_e = k \text{ and } m_{-k+2,k} \text{ is odd,} \\ \mathfrak{m} - \langle n_e+2, k\rangle + \delta_{n_e\neq k-2}\langle n_e+2, k-2\rangle & \text{if } -k+2 \leqslant n_e \leqslant k-2. \end{cases}$$

Remark 4.2. For $0 < k \in I$, the actions of $\widetilde{E}_{-k}$ and $\widetilde{F}_{-k}$ on $\mathfrak{m} \in \mathcal{M}_\theta$ are described by the following algorithm.

Step 1. Arrange segments in $\mathfrak{m}$ of the form $\langle -k, j\rangle$ $(j > k)$, $\langle -k+2, j\rangle$ $(j > k)$, $\langle i, k\rangle$ $(-k \leqslant i \leqslant$k), $\langle i, k-2\rangle$ $(-k+2 \leqslant i \leqslant k-2)$ in the order

$$\cdots, \langle -k, k+2\rangle, \langle -k+2, k+2\rangle,\ \langle -k, k\rangle, \langle -k+2, k\rangle, \langle -k+2, k-2\rangle, \\ \langle -k+4, k\rangle, \langle -k+4, k-2\rangle, \cdots, \langle k-2, k\rangle, \langle k-2, k-2\rangle, \langle k\rangle.$$

Step 2. Write signatures for each segment contained in $\mathfrak{m}$ by the following rules.

(i) If a segment is not $\langle -k+2, k\rangle$, then

- For $\langle -k, k\rangle$, write $--$,
- For $\langle -k, j\rangle$ with $j > k$, write $-$,
- For $\langle -k+2, k-2\rangle$ with $k > 1$, write $++$,
- For $\langle -k+2, j\rangle$ with $j > k$, write $+$,
- For $\langle j, k\rangle$ with $-k+2 < j \leqslant k$, write $-$,
- For $\langle j, k-2\rangle$ with $-k+2 < j \leqslant k-2$, write $+$,
- Otherwise, write no signature.

(ii) For segments $m_{-k+2,k}\langle -k+2, k\rangle$, if $m_{-k+2,k}$ is even, then write no signature, and if $m_{-k+2,k}$ is odd, then write $-+$.

Step 3. In the resulting sequence of $+$ and $-$, delete a subsequence of the form $+-$ and keep on deleting until no such subsequence remains.

Then we obtain a sequence of the form $--\cdots-++\cdots+$.

(1) $\varepsilon_{-k}(\mathfrak{m})$ is the total number of $-$ in the resulting sequence.

(2) $\widetilde{F}_{-k}(\mathfrak{m})$ is given as follows:

(i) if the leftmost $+$ corresponds to a segment $\langle -k+2, j\rangle$ for $j > k$, then replace it with $\langle -k, j\rangle$,

(ii) if the leftmost $+$ corresponds to a segment $\langle j, k-2\rangle$ for $-k+2 \leqslant j \leqslant k-2$, then replace it with $\langle j, k\rangle$,

(iii) if the leftmost $+$ corresponds to segment $m_{-k+2,k}\langle -k+2, k\rangle$, then replace one of the segments with $\langle -k, k\rangle$,

(iv) if no $+$ exists, add a segment $\langle k, k\rangle$ to $\mathfrak{m}$.

(3) $\widetilde{E}_{-k}(\mathfrak{m})$ is given as follows:

(i) if the rightmost $-$ corresponds to a segment $\langle -k, j\rangle$ for $j \geqslant k$, then replace it with $\langle -k+2, j\rangle$,

(ii) if the rightmost $-$ corresponds to a segment $\langle j, k\rangle$ for $-k+2 < j < k$, then replace it with $\langle j, k-2\rangle$,

(iii) if the rightmost $-$ corresponds to segments $m_{-k+2,k}\langle -k+2, k\rangle$, then replace one of the segment with $\langle -k+2, k-2\rangle$,

(iv) if the rightmost $-$ corresponds to a segment $\langle k, k\rangle$ for $k > 1$, then delete it,

(v) if no $-$ exists, then $\widetilde{E}_{-k}(\mathfrak{m}) = 0$.

Example 4.3. (1) We shall write $\{a, b\}$ for $a\langle -1, 1\rangle + b\langle 1\rangle$. The following diagram is the part of the crystal graph of $\mathrm{B}_\theta(0)$ that concerns only the 1-arrows and the (-1)-arrows.

$$\begin{tikzcd}
 & & & & \{0,4\} \arrow[r, shift left, "1"] \arrow[r, shift right, "-1"'] & \{0,5\}\cdots \\
 & & \{0,2\} \arrow[r, shift left, "1"] \arrow[r, shift right, "-1"'] & \{0,3\} \arrow[ur, "1"] \arrow[dr, "-1"] & & \\
\phi \arrow[r, shift left, "1"] \arrow[r, shift right, "-1"'] & \{0,1\} \arrow[ur, "1"] \arrow[dr, "-1"'] & & & \{1,2\} \arrow[r, shift left, "1"] \arrow[r, shift right, "-1"'] & \{1,3\}\cdots \\
 & & \{1,0\} \arrow[r, shift left, "1"] \arrow[r, shift right, "-1"'] & \{1,1\} \arrow[ur, "1"] \arrow[dr, "-1"'] & & \\
 & & & & \{2,0\} \arrow[r, shift left, "1"] \arrow[r, shift right, "-1"'] & \{2,1\}\cdots
\end{tikzcd}$$

Especially the part of (-1)-arrows is the following diagram.

$$\{0, 2n\} \xrightarrow{-1} \{0, 2n+1\} \xrightarrow{-1} \{1, 2n\} \xrightarrow{-1} \{1, 2n+1\} \xrightarrow{-1} \{2, 2n\} \cdots\cdots$$

(2) The following diagram is the part of the crystal graph of $\mathrm{B}_\theta(0)$ that concerns only the (-1)-arrows and the (-3)-arrows. This diagram is, as a graph, isomorphic to the crystal graph of A_2.

$$\begin{tikzcd}[column sep=small, row sep=tiny]
 & & & & 2\langle -1,1\rangle \\
 & & & \langle -1,1\rangle + \langle 1\rangle \arrow[ur, "-1"] \arrow[dr, "-3"'] & \\
 & & \langle -1,1\rangle \arrow[ur, "-1"] \arrow[dr, "-3"'] & & \langle -1,3\rangle + \langle 1\rangle \\
 & \langle 1\rangle \arrow[ur, "-1"] \arrow[rdd, "-3"'] & & \langle -1,3\rangle \arrow[ur, "-1"'] \arrow[dr, "-3"'] & \\
 & & & & \langle -3,3\rangle \\
 & & \langle 1,3\rangle \arrow[ruu, "-1"] \arrow[rdddd, "-3"] & & \\
 & & & & \langle 3\rangle + \langle -1,1\rangle + \langle 1\rangle \\
 & & & \langle 3\rangle + \langle -1,1\rangle \arrow[ur, "-1"] \arrow[dr, "-3"'] & \\
\phi \arrow[ruuuuu, "-1"] \arrow[rddddd, "-3"'] & & & & \langle -1,3\rangle + \langle 3\rangle \\
 & & & \langle 1,3\rangle + \langle 3\rangle \arrow[ur, "-1"'] \arrow[dr, "-3"'] & \\
 & & & & \langle 1,3\rangle + 2\langle 3\rangle \\
 & & \langle 3\rangle + \langle 1\rangle \arrow[ruuuu, "-1"] \arrow[rdd, "-3"'] & & \\
 & & & & 2\langle 3\rangle + \langle -1,1\rangle \\
 & \langle 3\rangle \arrow[ruu, "-1"] \arrow[dr, "-3"'] & & 2\langle 3\rangle + \langle 1\rangle \arrow[ur, "-1"'] \arrow[dr, "-3"'] & \\
 & & 2\langle 3\rangle \arrow[ur, "-1"] \arrow[dr, "-3"'] & & 3\langle 3\rangle + \langle 1\rangle \\
 & & & 3\langle 3\rangle \arrow[ur, "-1"'] \arrow[dr, "-3"'] & \\
 & & & & 4\langle 3\rangle
\end{tikzcd}$$

(3) Here is the part of the crystal graph of $\mathrm{B}_\theta(0)$ that concerns only the n-arrows and the $(-n)$-arrows for an odd integer $n \geqslant 3$:

$$\begin{tikzcd}
\phi \arrow[r, shift left, "n"] \arrow[r, shift right, "-n"'] & \langle n\rangle \arrow[r, shift left, "n"] \arrow[r, shift right, "-n"'] & 2\langle n\rangle \arrow[r, shift left, "n"] \arrow[r, shift right, "-n"'] & 3\langle n\rangle \arrow[r, shift left, "n"] \arrow[r, shift right, "-n"'] & \cdots\cdots
\end{tikzcd}$$

Lemma 4.4. *For $k \in I_{>0}$, the data $\widetilde{E}_{-k}$, $\widetilde{F}_{-k}$, ε_{-k} define a crystal structure on $\mathcal{M}_\theta$, namely we have*

(i) $\widetilde{F}_{-k}\mathcal{M}_\theta \subset \mathcal{M}_\theta$ *and* $\widetilde{E}_{-k}\mathcal{M}_\theta \subset \mathcal{M}_\theta \sqcup \{0\}$,

(ii) $\widetilde{F}_{-k}\widetilde{E}_{-k}(\mathfrak{m}) = \mathfrak{m}$ *if* $\widetilde{E}_{-k}(\mathfrak{m}) \neq 0$, *and* $\widetilde{E}_{-k} \circ \widetilde{F}_{-k} = \mathrm{id}$,

(iii) $\varepsilon_{-k}(\mathfrak{m}) = \max\left\{n \geqslant 0 \mid \widetilde{E}^n_{-k}(\mathfrak{m}) \neq 0\right\}$ *for any* $\mathfrak{m} \in \mathcal{M}_\theta$.

Proof. We shall first show that, for $\mathfrak{m} = \sum_{-j \leqslant i \leqslant j} m_{i,j}\langle i, j\rangle \in \mathcal{M}_\theta$, $\widetilde{F}_{-k}(\mathfrak{m})$ is θ-restricted, $\widetilde{E}_{-k}\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m}$ and $\varepsilon_{-k}(\widetilde{F}_{-k}\mathfrak{m}) = \varepsilon_{-k}(\mathfrak{m}) + 1$. Let $A_j := A_j^{(-k)}(\mathfrak{m})$ $(j \geqslant -k+2)$ and let n_f be as in Definition 4.1. Set $\mathfrak{m}' = \widetilde{F}_{-k}\mathfrak{m}$. Let $A'_j = A_j^{(-k)}(\mathfrak{m}')$ and let n'_e be n_e for $\mathfrak{m}'$.

(i) Assume $n_f > k$. Since $A_{n_f} > A_{n_f-2} = A_{n_f} + m_{-k,n_f-2} - m_{-k+2,n_f}$, we have $m_{-k,n_f-2} < m_{-k+2,n_f}$. Hence $\mathfrak{m}' = \mathfrak{m} - \langle -k+2, n_f\rangle + \langle -k, n_f\rangle$ is θ-restricted. Then we have

$$A'_j = \begin{cases} A_j & \text{if } j > n_f, \\ A_j + 1 & \text{if } j = n_f, \\ A_j + 2 & \text{if } j < n_f. \end{cases}$$

Hence $\varepsilon_{-k}(\mathfrak{m}') = A_{n_f} + 1 = \varepsilon_{-k}(\mathfrak{m}) + 1$ and $n'_e = n_f$, which implies $\mathfrak{m} = \widetilde{E}_{-k}(\mathfrak{m}')$.

(ii) Assume $n_f = k$.

(a) If $m_{-k+2,k}$ is odd, then $\mathfrak{m}' = \mathfrak{m} - \langle -k+2, k\rangle + \langle -k, k\rangle$ is θ-restricted. We have

$$A'_j = \begin{cases} A_j & \text{if } j > k, \\ A_j + 1 & \text{if } j = k, \\ A_j + 2 & \text{if } j < k, \end{cases}$$

Hence $\varepsilon_{-k}(\mathfrak{m}') = \varepsilon_{-k}(\mathfrak{m}) + 1$ and $n'_e = k$, which implies $\mathfrak{m} = \widetilde{E}_{-k}(\mathfrak{m}')$.

(b) Assume that $m_{-k+2,k}$ is even. If $k \neq 1$, then $A_k > A_{-k+2} = A_k - 2m_{-k+2,k-2}$, and hence $m_{-k+2,k-2} > 0$. Therefore $\mathfrak{m}' = \mathfrak{m} - \delta_{k\neq 1}\langle -k+2, k-2\rangle + \langle -k+2, k\rangle$ is θ-restricted. We have

$$A'_j = \begin{cases} A_j & \text{if } j > k, \\ A_j + 1 & \text{if } j = k, \\ A_j + 2 & \text{if } j < k. \end{cases}$$

Hence $\varepsilon_{-k}(\mathfrak{m}') = \varepsilon_{-k}(\mathfrak{m}) + 1$ and $n'_e = k$, which implies $\mathfrak{m} = \widetilde{E}_{-k}(\mathfrak{m}')$.

(iii) Assume $-k+2 \leqslant n_f < k-2$. Since $A_{n_f} > A_{n_f+2} = A_{n_f} + m_{n_f+4,k} - m_{n_f+2,k-2}$, we have $m_{n_f+2,k-2} > m_{n_f+4,k}$. Hence $\mathfrak{m}' = \mathfrak{m} - \langle n_f+2, k-2\rangle + \langle n_f+2, k\rangle$ is θ-restricted. Then we have

$$A'_j = \begin{cases} A_j & \text{if } j > n_f, \\ A_j + 1 & \text{if } j = n_f, \\ A_j + 2 & \text{if } j < n_f. \end{cases}$$

(Here the ordering is as in Definition 4.1 (i).) Hence $\varepsilon_{-k}(\mathfrak{m}') = \varepsilon_{-k}(\mathfrak{m}) + 1$ and $n'_e = n_f$, which implies $\mathfrak{m} = \widetilde{E}_{-k}\mathfrak{m}'$.

(iv) Assume $n_f = k-2$. It is obvious that $\mathfrak{m}' = \mathfrak{m} + \langle k\rangle$ is θ-restricted. We have

$$A'_j = \begin{cases} A_j & \text{if } j \neq n_f, \\ A_j + 1 & \text{if } j = n_f. \end{cases}$$

Hence $\varepsilon_{-k}(\mathfrak{m}') = \varepsilon_{-k}(\mathfrak{m}) + 1$ and $n'_e = n_f$, which implies $\mathfrak{m} = \widetilde{E}_{-k}(\mathfrak{m}')$.

Similarly, we can prove that if $\varepsilon_{-k}(\mathfrak{m}) > 0$, then $\widetilde{E}_{-k}(\mathfrak{m})$ is θ-restricted and $\widetilde{F}_{-k}\widetilde{E}_{-k}(\mathfrak{m}) = \mathfrak{m}$. Hence we obtain the desired results. Q.E.D.

Definition 4.5. *For $k \in I_{>0}$, we define $\widetilde{F}_k$, $\widetilde{E}_k$ and ε_k by the same rule as in* Definition 3.7 *for $\tilde{f}_k$, $\tilde{e}_k$ and ε_k.*

Since it is well-known that it gives a crystal structure on $\mathcal{M}$, we obtain the following result.

Theorem 4.6. *By $\widetilde{F}_k$, $\widetilde{E}_k$, ε_k $(k \in I)$, $\mathcal{M}_\theta$ is a crystal, namely, we have*

(i) $\widetilde{F}_k\mathcal{M}_\theta \subset \mathcal{M}_\theta$ *and* $\widetilde{E}_k\mathcal{M}_\theta \subset \mathcal{M}_\theta \sqcup \{0\}$,

(ii) $\widetilde{F}_k\widetilde{E}_k(\mathfrak{m}) = \mathfrak{m}$ *if* $\widetilde{E}_k(\mathfrak{m}) \neq 0$, *and* $\widetilde{E}_k \circ \widetilde{F}_k = \mathrm{id}$,

(iii) $\varepsilon_k(\mathfrak{m}) = \max\left\{n \geqslant 0 \mid \widetilde{E}_k^n(\mathfrak{m}) \neq 0\right\}$ *for any* $\mathfrak{m} \in \mathcal{M}_\theta$.

The crystal $\mathcal{M}_\theta$ has a unique highest weight vector.

Lemma 4.7. *If $\mathfrak{m} \in \mathcal{M}_\theta$ satisfies that $\varepsilon_k(\mathfrak{m}) = 0$ for any $k \in I$, then $\mathfrak{m} = \emptyset$. Here $\emptyset$ is the empty multisegment. In particular, for any $\mathfrak{m} \in \mathcal{M}_\theta$, there exist $\ell \geqslant 0$ and $i_1, \ldots, i_\ell \in I$ such that $\mathfrak{m} = \widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell}\emptyset$.*

Proof. Assume $\mathfrak{m} \neq \emptyset$. Let k be the largest k such that $m_{k,j} \neq 0$ for some j. Then take the largest j such that $m_{k,j} \neq 0$. Then $j \geqslant |k|$. Moreover, we have $m_{k+2,\ell} = 0$ for any ℓ, and $m_{k,\ell} = 0$ for any $\ell > j$. Hence we have

$$A_j^{(k)}(\mathfrak{m}) = \begin{cases} 2m_{k,j} & \text{if } k = -j, \\ m_{k,j} & \text{otherwise.} \end{cases}$$

Hence $\varepsilon_k(\mathfrak{m}) \geqslant A_j^{(k)}(\mathfrak{m}) > 0$. Q.E.D.

4.2. **A criterion for crystals.** We shall give a criterion for a basis to be a crystal basis. Although we treat the case for modules over $\mathcal{B}(\mathfrak{g})$ in this paper, similar results hold also for $U_q(\mathfrak{g})$.

Let $\mathbf{K}[e, f]$ be the ring generated by e and f with the defining relation $ef = q^{-2}fe + 1$. We define the divided power by $f^{(n)} = f^n/[n]!$.

Let P be a free $\mathbb{Z}$-module, and let α be a non-zero element of P.

Let M be a $\mathbf{K}[e, f]$-module. Assume that M has a weight decomposition $M = \oplus_{\xi \in P} M_\xi$, and $eM_\lambda \subset M_{\lambda+\alpha}$ and $fM_\lambda \subset M_{\lambda-\alpha}$.

Assume the following finiteness conditions:

$$\text{for any } \lambda \in P,\ \dim M_\lambda < \infty \text{ and } M_{\lambda+n\alpha} = 0 \text{ for } n \gg 0. \tag{4.1}$$

Hence for any $u \in M$, we can write $u = \sum_{n \geqslant 0} f^{(n)}u_n$ with $eu_n = 0$. We define endomorphisms $\tilde{e}$ and $\tilde{f}$ of M by

$$\tilde{e}u = \sum_{n \geqslant 1} f^{(n-1)}u_n,$$
$$\tilde{f}u = \sum_{n \geqslant 0} f^{(n+1)}u_n.$$

Let B be a crystal with weight decomposition by P. In this paper, we consider only the following type of crystals. We have $\mathrm{wt}\colon B \to P$, $\tilde{f}\colon B \to B$, $\tilde{e}\colon B \to B \sqcup \{0\}$, $\varepsilon\colon B \to \mathbb{Z}_{\geqslant 0}$ satisfying the following properties, where $B_\lambda := \mathrm{wt}^{-1}(\lambda)$:

(i) $\tilde{f}B_\lambda \subset B_{\lambda-\alpha}$ and $\tilde{e}B_\lambda \subset B_{\lambda+\alpha} \sqcup \{0\}$ for any $\lambda \in P$,

(ii) $\tilde{f}\tilde{e}(b) = b$ if $\tilde{e}b \neq 0$, and $\tilde{e} \circ \tilde{f} = \mathrm{id}_B$,
(iii) for any $\lambda \in P$, B_λ is a finite set and $B_{\lambda+n\alpha} = \emptyset$ for $n \gg 0$,
(iv) $\varepsilon(b) = \max\{n \geqslant 0 \mid \tilde{e}^n b \neq 0\}$ for any $b \in B$.

Set $\mathrm{ord}(a) = \sup\{n \in \mathbb{Z} \mid a \in q^n \mathbf{A}_0\}$ for $a \in \mathbf{K}$. We understand $\mathrm{ord}(0) = \infty$.

Let $\{C(b)\}_{b\in B}$ be a system of generators of M with $C(b) \in M_{\mathrm{wt}(b)}$: $M = \sum_{b\in B} \mathbf{K}C(b)$.

Let ξ be a map from B to an ordered set. Let $c\colon \mathbb{Z} \to \mathbb{R}$, $f\colon \mathbb{Z} \to \mathbb{R}$ and $e\colon \mathbb{Z} \to \mathbb{R}$. Assume that a decomposition $B = B' \cup B''$ is given.

Assume that we have expressions:

$$eC(b) = \sum_{b'\in B} E_{b,b'} C(b'), \tag{4.2}$$

$$fC(b) = \sum_{b'\in B} F_{b,b'} C(b'). \tag{4.3}$$

Now consider the following conditions for these data, where $l = \varepsilon(b)$ and $l' = \varepsilon(b')$:

(4.4) $c(0) = 0$, and $c(n) > 0$ for $n \neq 0$,
(4.5) $c(n) \leqslant n + c(m+n) + e(m) \quad$ for $n \geqslant 0$,
(4.6) $c(n) \leqslant c(m+n) + f(m) \quad$ for $n \leqslant 0$,
(4.7) $c(n) + f(n) > 0 \quad$ for $n > 0$,
(4.8) $c(n) + e(n) > 0 \quad$ for $n > 0$,
(4.9) $\mathrm{ord}(F_{b,b'}) \geqslant -\ell + f(\ell + 1 - \ell')$,
(4.10) $\mathrm{ord}(E_{b,b'}) \geqslant 1 - \ell + e(\ell - 1 - \ell')$,
(4.11) $F_{b,\tilde{f}b} \in q^{-\ell}(1 + q\mathbf{A}_0)$,
(4.12) $E_{b,\tilde{e}b} \in q^{1-\ell}(1 + q\mathbf{A}_0) \quad$ if $\ell > 0$,
(4.13) $\mathrm{ord}(F_{b,b'}) > -\ell + f(\ell + 1 - \ell') \quad$ if $b' \neq \tilde{f}b$, $\xi(\tilde{f}b) \not> \xi(b')$,
(4.14) $\mathrm{ord}(F_{b,b'}) > -\ell + f(\ell + 1 - \ell')$ if $\tilde{f}b \in B'$, $b' \neq \tilde{f}b$ and $\ell \leqslant \ell' - 1$,
(4.15) $\mathrm{ord}(E_{b,b'}) > 1 - \ell + e(\ell - 1 - \ell')$ if $b \in B''$, $b' \neq \tilde{e}b$ and $\ell \leqslant \ell' + 1$.

Theorem 4.8. *Assume the conditions* (4.4)–(4.15)*. Set $L = \sum_{b\in B} \mathbf{A}_0 C(b)$. Then we have $\tilde{e}L \subset L$ and $\tilde{f}L \subset L$. Moreover we have*

$$\tilde{e}C(b) \equiv C(\tilde{e}b) \bmod qL \quad \textit{and} \quad \tilde{f}C(b) \equiv C(\tilde{f}b) \bmod qL \quad \textit{for any } b \in B.$$

Here we understand $C(0) = 0$.

We shall divide the proof into several steps.

Write

$$C(b) = \sum_{n\geqslant 0} f^{(n)} C_n(b) \quad \text{with } eC_n(b) = 0.$$

Set

$$L_0 = \sum_{b\in B,\, n\geqslant 0} \mathbf{A}_0 f^{(n)} C_0(b).$$

Set for $u \in M$, $\mathrm{ord}(u) = \sup\{n \in \mathbb{Z} \mid u \in q^n L_0\}$. If $u = 0$ we set $\mathrm{ord}(u) = \infty$, and if $u \notin \cup_{n\in\mathbb{Z}} q^n L_0$, then $\mathrm{ord}(u) = -\infty$.

We shall use the following two recursion formulas (4.16) and (4.17).

We have

$$\begin{aligned} eC(b) &= \sum_{n\geqslant 1} q^{1-n} f^{(n-1)} C_n(b) \\ &= \sum_{n\geqslant 0} E_{b,b'} f^{(n)} C_n(b'). \end{aligned}$$

Hence we have

$$C_n(b) = \sum_{b'\in B_{\lambda+\alpha}} q^{n-1} E_{b,b'} C_{n-1}(b') \quad \text{for } n>0 \text{ and } b\in B_\lambda. \tag{4.16}$$

If $\ell := \varepsilon(b) > 0$, then we have

$$\begin{aligned} fC(\tilde{e}b) &= \sum_{b'\in B,\ n\geqslant 0} F_{\tilde{e}b,b'} f^{(n)} C_n(b') \\ &= \sum_{n\geqslant 0} [n+1] f^{(n+1)} C_n(\tilde{e}b). \end{aligned}$$

Hence, we have by (4.11)

$$\begin{aligned} \delta_{n\neq 0}[n] C_{n-1}(\tilde{e}b) &= \textstyle\sum_{b'} F_{\tilde{e}b,b'} C_n(b') \\ &\in q^{1-\ell}(1+q\mathbf{A}_0) C_n(b) + \textstyle\sum_{b'\neq b} F_{\tilde{e}b,b'} C_n(b'). \end{aligned}$$

Therefore we obtain

$$C_n(b) \in \delta_{n\neq 0}(1+q\mathbf{A}_0) q^{\ell-n} C_{n-1}(\tilde{e}b) + \sum_{b'\neq b} q^{\ell-1} \mathbf{A}_0 F_{\tilde{e}b,b'} C_n(b') \quad \text{if } \ell>0. \tag{4.17}$$

Lemma 4.9. $\operatorname{ord}(C_n(b)) \geqslant c(n-\ell)$ *for any* $n\in\mathbb{Z}_{\geqslant 0}$ *and* $b\in B$*, where* $\ell := \varepsilon(b)$.

Proof. For $\lambda\in P$, we shall show the assertion for $b\in B_\lambda$ by the induction on $\sup\{n\in\mathbb{Z} \mid M_{\lambda+n\alpha}\neq 0\}$. Hence we may assume

$$\operatorname{ord}(C_n(b)) \geqslant c(n-\ell) \text{ for any } n\in\mathbb{Z}_{\geqslant 0} \text{ and } b\in B_{\lambda+\alpha}. \tag{4.18}$$

(i) Let us first show $C_n(b)\in\mathbf{K}L_0$.

Since it is trivial for $n=0$, assume that $n>0$. Since $C_{n-1}(b')\in\mathbf{K}L_0$ for $b'\in B_{\lambda+\alpha}$ by the induction assumption (4.18), we have $C_n(b)\in\mathbf{K}L_0$ by (4.16).

(ii) Let us show that $\operatorname{ord}(C_n(b)) \geqslant c(n-\ell)$ for $n\geqslant\ell$.

If $n=0$, then $\ell=0$ and the assertion is trivial by (4.4). Hence we may assume that $n>0$.

We shall use (4.16). For $b'\in B_{\lambda+\alpha}$, we have

$$\operatorname{ord}(C_{n-1}(b')) \geqslant c(n-1-\ell') \quad \text{where } \ell'=\varepsilon(b')$$

by the induction hypothesis (4.18). On the other hand, $\operatorname{ord}(E_{b,b'}) \geqslant 1-\ell+e(\ell-1-\ell')$ by (4.10). Hence,

$$\begin{aligned} \operatorname{ord}(q^{n-1}E_{b,b'}C_{n-1}(b')) &\geqslant (n-1) + \big(1-\ell+e(\ell-1-\ell')\big) + c(n-1-\ell') \\ &= (n-\ell) + e(\ell-1-\ell') + c((n-\ell)+(\ell-1-\ell')) \\ &\geqslant c(n-\ell) \end{aligned}$$

by (4.5).

(iii) In the general case, let us set $r = \min\{\operatorname{ord}(C_n(b)) - c(n-\varepsilon(b)) \mid b\in B_\lambda,\ n\geqslant 0\} \in \mathbb{R}\cup\{\infty\}$. Assuming $r<0$, we shall prove

$$\operatorname{ord}(C_n(b)) > c(n-\ell) + r \quad \text{for any } b\in B_\lambda,$$

which leads a contradiction.

By the induction on $\xi(b)$, we may assume that

$$\text{if } \xi(b') < \xi(b), \text{ then } \operatorname{ord}(C_n(b')) > c(n-\ell') + r \text{ where } \ell' := \varepsilon(b'). \tag{4.19}$$

By (ii), we may assume that $n < \ell$. Hence $\tilde{e}b \in B$. By the induction hypothesis (4.18), we have $\operatorname{ord}(q^{\ell-n}C_{n-1}(\tilde{e}b)) \geqslant \ell - n + c((n-1)-(\ell-1)) \geqslant c(n-\ell) > c(n-\ell)+r$. By (4.17), it is enough to show

$$\operatorname{ord}(q^{\ell-1}F_{\tilde{e}b,b'}C_n(b')) > c(n-\ell)+r \quad \text{for } b' \neq b.$$

We shall divide its proof into two cases.

(a) $\xi(b') < \xi(b)$.

In this case, (4.19) implies $\operatorname{ord}(C_n(b')) > c(n-\ell')+r$. Hence

$$\begin{aligned} \operatorname{ord}(q^{\ell-1}F_{\tilde{e}b,b'}C_n(b')) &> (\ell-1)+(1-\ell+f(\ell-\ell'))+c(n-\ell')+r \\ &= f(\ell-\ell')+c((n-\ell)+(\ell-\ell'))+r \geqslant c(n-\ell)+r \end{aligned}$$

by (4.9) and (4.6).

(b) Case $\xi(b') \not< \xi(b)$.

In this case, $\operatorname{ord}(F_{\tilde{e}b,b'}) > 1-\ell+f(\ell-\ell')$ by (4.13), and $\operatorname{ord}(C_n(b')) \geqslant c(n-\ell')+r$. Hence,

$$\begin{aligned} \operatorname{ord}(q^{\ell-1}F_{\tilde{e}b,b'}C_n(b')) &> (\ell-1)+(1-\ell+f(\ell-\ell'))+c(n-\ell')+r \\ &= f(\ell-\ell')+c((n-\ell)+(\ell-\ell'))+r \geqslant c(n-\ell)+r. \end{aligned}$$

Q.E.D.

Lemma 4.10. $\operatorname{ord}(C_\ell(b) - C_{\ell-1}(\tilde{e}b)) > 0$ *for* $\ell := \varepsilon(b) > 0$.

Proof. We divide the proof into two cases: $b \in B'$ and $b \in B''$.

(i) $b \in B'$.

By (4.17), it is enough to show

$$\operatorname{ord}(q^{\ell-1}F_{\tilde{e}b,b'}C_\ell(b')) > 0 \quad \text{for } b' \neq b.$$

(a) Case $\ell > \ell' := \varepsilon(b')$.

We have

$$\operatorname{ord}(q^{\ell-1}F_{\tilde{e}b,b'}C_\ell(b')) \geqslant (\ell-1)+(1-\ell+f(\ell-\ell'))+c(\ell-\ell') > 0$$

by (4.7).

(b) Case $\ell \leqslant \ell'$.

We have $\operatorname{ord}(F_{\tilde{e}b,b'}) > 1-\ell+f(\ell-\ell')$ by (4.14). Hence

$$\operatorname{ord}(q^{\ell-1}F_{\tilde{e}b,b'}C_\ell(b')) > (\ell-1)+(1-\ell+f(\ell-\ell'))+c(\ell-\ell') \geqslant 0$$

by (4.6) with $n=0$.

(ii) Case $b \in B''$.

We use (4.16). By (4.12), it is enough to show that

$$\operatorname{ord}(q^{\ell-1}E_{b,b'}C_{\ell-1}(b')) > 0 \text{ for } b' \neq \tilde{e}b.$$

(a) Case $\ell - 1 > \ell'$.

$\operatorname{ord}(q^{\ell-1}E_{b,b'}C_{\ell-1}(b')) \geqslant e(\ell-1-\ell')+c(\ell-1-\ell') > 0$ by (4.10) and (4.8).

(b) Case $\ell - 1 \leqslant \ell'$.

$\operatorname{ord}(E_{b,b'}) > 1-\ell+e(\ell-1-\ell')$ by (4.15), and $\operatorname{ord}(q^{\ell-1}E_{b,b'}C_{\ell-1}(b')) > e(\ell-1-\ell')+c(\ell-1-\ell') \geqslant 0$ by (4.5) with $n=0$.

Q.E.D.

Hence we have

$$\begin{aligned}
&C_n(b) \equiv 0 \bmod qL_0 \quad \text{for } n \neq \ell := \varepsilon(b),\\
&C_\ell(b) \equiv C_0(\tilde{e}^\ell b) \bmod qL_0,\\
&C(b) \equiv f^{(\ell)} C_\ell(b) \bmod qL_0,\\
&\tilde{f}C(b) \equiv C(\tilde{f}b) \bmod qL_0,\\
&\tilde{e}C(b) \equiv C(\tilde{e}b) \bmod qL_0,\\
&L_0 := \sum_{b\in B,\, n\geqslant 0} \mathbf{A}_0 f^{(n)} C_0(b) = \sum_{b\in B} \mathbf{A}_0 C(b).
\end{aligned}$$

Indeed, the last equality follows from the fact that $\{C(b)\}_{b\in B}$ generates L_0/qL_0.

Thus we have completed the proof of Theorem 4.8.

The following is the special case where $B' = B'' = B$ and $\xi(b) = \varepsilon(b)$.

Corollary 4.11. *Assume* (4.4)–(4.12) *and*

$$\operatorname{ord}(F_{b,b'}) > -\ell + f(1+\ell-\ell') \quad \text{if } \ell < \ell' \text{ and } b' \neq \tilde{f}b, \tag{4.20}$$

$$\operatorname{ord}(E_{b,b'}) > 1-\ell + e(\ell-1-\ell') \quad \text{if } \ell \leqslant \ell'+1 \text{ and } b' \neq \tilde{e}b. \tag{4.21}$$

Then the assertions of Theorem 4.8 *hold.*

4.3. **Estimates of the order of coefficients.** By applying Theorem 4.8, we shall show that $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ is a crystal basis of $V_\theta(0)$ and its crystal structure coincides with the one given in § 4.1.

We define $c, f, e\colon \mathbb{Z} \to \mathbb{Q}$ by $c(n) = \lfloor n/2 \rfloor$ and $f(n) = e(n) = n/2$. Then the conditions (4.4)–(4.8) are obvious. Set $\xi(\mathfrak{m}) = (-1)^{m_{-k+2,k}} m_{-k,k}$ and

$$\begin{aligned}
B'' &= \{\mathfrak{m} \in \mathcal{M}_\theta \mid -k+2 \leqslant n_e(\mathfrak{m}) < k\} \cup \{\mathfrak{m} \in \mathcal{M}_\theta \mid m_{-k+2,k}(\mathfrak{m}) \text{ is odd}\},\\
B' &= \mathcal{M}_\theta \setminus B''.
\end{aligned}$$

Here $n_e(\mathfrak{m})$ is n_e given in Definition 4.1 (ii). If $\varepsilon_{-k}(\mathfrak{m}) = 0$, then we understand $n_e(\mathfrak{m}) = \infty$.

We define $F^{-k}_{\mathfrak{m},\mathfrak{m}'}$ and $E^{-k}_{\mathfrak{m},\mathfrak{m}'}$ by the coefficients of the following expansion:

$$\begin{aligned}
F_{-k} P_\theta(\mathfrak{m})\widetilde{\phi} &= \sum_{\mathfrak{m}'} F^{-k}_{\mathfrak{m},\mathfrak{m}'} P_\theta(\mathfrak{m}')\widetilde{\phi},\\
E_{-k} P_\theta(\mathfrak{m})\widetilde{\phi} &= \sum_{\mathfrak{m}'} E^{-k}_{\mathfrak{m},\mathfrak{m}'} P_\theta(\mathfrak{m}')\widetilde{\phi},
\end{aligned}$$

as given in Theorems 3.21 and 3.22. Put $\ell = \varepsilon_{-k}(\mathfrak{m})$ and $\ell' = \varepsilon_{-k}(\mathfrak{m}')$.

Proposition 4.12. *The conditions* (4.9), (4.11), (4.13) *and* (4.14) *hold, namely, we have*

(a) *if* $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, *then* $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-\ell}(1+q\mathbf{A}_0)$,

(b) *if* $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$, *then* $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) \geqslant -\ell + f(\ell+1-\ell') = -(\ell+\ell'-1)/2$,

(c) *if* $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$ *and* $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, *then the following two conditions hold:*
   (1) $\xi(\widetilde{F}_{-k}(\mathfrak{m})) > \xi(\mathfrak{m}')$,
   (2) $\ell \geqslant \ell'$ *or* $\widetilde{F}_{-k}(\mathfrak{m}) \in B''$.

Proof. We shall write A_j for $A^{-k}_j(\mathfrak{m})$. Let n_f be as in Definition 4.1 (i).

Note that $F^{-k}_{\mathfrak{m},\widetilde{F}_{-k}(\mathfrak{m})} \neq 0$.

If $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \neq 0$, we have the following four cases. We shall use $[n] \in q^{1-n}(1+q\mathbf{A}_0)$ for $n > 0$.

Case 1. $\mathfrak{m}' = \mathfrak{m} - \langle -k+2, n\rangle + \langle -k, n\rangle$ for $n > k$.

In this case, we have

$$F^{-k}_{\mathfrak{m},\mathfrak{m}'} = [m_{-k,n}+1]q^{\sum_{j>n}(m_{-k+2,j}-m_{-k,j})} \in q^{-A_n}(1+q\mathbf{A}_0)$$

and

$$\ell = \max\{A_j (j \geqslant -k+2)\},$$
$$\ell' = \max\{A_j\ (j > n), A_n+1, A_j+2\ (j < n)\}.$$

If $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, then $\ell = A_n$ and we obtain (a). Assume $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$. Since $A_n \leqslant \ell, \ell'-1$, we have $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_n \geqslant -(\ell+\ell'-1)/2$. Hence we obtain (b). If $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then we have $A_n = \ell = \ell'-1$. Since $A_j + 2 \leqslant \ell' = A_n+1$ for $j < n$, we have $n_f = n$ and $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, which is a contradiction.

Case 2. $\mathfrak{m}' = \mathfrak{m} - \langle -k+2, k\rangle + \langle -k, k\rangle$.

In this case we have

$$F^{-k}_{\mathfrak{m},\mathfrak{m}'} = [2m_{-k,k}+2]q^{\sum_{j>k}(m_{-k+2,j}-m_{-k,j})} \in q^{-A_k-\delta(\mathfrak{m}_{-k+2,k} \text{ is even})}(1+q\mathbf{A}_0).$$

(i) Assume that $m_{-k+2,k}$ is odd. We have $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k}(1+q\mathbf{A}_0)$ and

$$\ell' = \max\{A_j\ (j > k), A_k+1, A_j+2\ (j < k)\}.$$

If $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, then $\ell = A_k$ and (a) holds. Assume that $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$. We have $A_k \leqslant \ell, \ell'-1$ and hence $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_k \geqslant -(\ell+\ell'-1)/2$. If $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k = \ell = \ell'-1$, and we have $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, which is a contradiction.

(ii) Assume that $m_{-k+2,k}$ is even. Then $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$, $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k-1}(1+q\mathbf{A}_0)$ and

$$\ell' = \max\{A_j\ (j > k), A_k+3, A_j+2\ (j < k)\}.$$

We have $A_k \leqslant \ell, \ell'-3$ and hence $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_k - 1 \geqslant -(\ell+\ell'-1)/2$. Hence (b) holds. Let us show (c). Assume $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$, and $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$. Then we have $A_k = \ell = \ell'-3$. Hence $n_f \leqslant k$ and we have either $\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m} - \delta_{i\neq k}\langle i, k-2\rangle + \langle i, k\rangle$ with $-k+2 < i \leqslant k$ or $\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m} - \delta_{k\neq 1}\langle -k+2, k-2\rangle + \langle -k+2, k\rangle$. Hence we have $\xi(\widetilde{F}_{-k}(\mathfrak{m})) = \pm m_{-k,k} > -m_{-k,k} - 1 = \xi(\mathfrak{m}')$. Hence we obtain (c) (1).

(1) Assume $\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m} - \delta_{i\neq k}\langle i, k-2\rangle + \langle i, k\rangle$ with $-k+2 < i \leqslant k$. Then $k \neq 1$ and $\widetilde{E}_{-k}(\widetilde{F}_{-k}(\mathfrak{m})) = \widetilde{F}_{-k}(\mathfrak{m}) - \langle i, k\rangle + \delta_{i\neq k}\langle i, k-2\rangle$. Hence $n_e(\widetilde{F}_{-k}(\mathfrak{m})) = i - 2 < k$. Hence $\widetilde{F}_{-k}(\mathfrak{m}) \in B''$. Therefore we obtain (c) (2).

(2) Assume $\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m} - \delta_{k\neq 1}\langle -k+2, k-2\rangle + \langle -k+2, k\rangle$. Then $m_{-k+2,k}(\widetilde{F}_{-k}(\mathfrak{m})) = m_{-k+2,k}+1$ is odd. Hence $\widetilde{F}_{-k}(\mathfrak{m}) \in B''$.

Case 3. $\mathfrak{m}' = \mathfrak{m} - \delta_{k\neq 1}\langle -k+2, k-2\rangle + \langle -k+2, k\rangle$. In this case, we have

$$F^{-k}_{\mathfrak{m},\mathfrak{m}'} = [m_{-k+2,k}+1]q^{\sum_{j>k}(m_{-k+2,j}-m_{-k,j})+m_{-k+2,k}-2m_{-k,k}} \in q^{-A_k+\delta(m_{-k+2,k} \text{ is odd})}(1+q\mathbf{A}_0).$$

(i) If $m_{-k+2,k}$ is odd, then $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$, $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k+1}(1+q\mathbf{A}_0)$, and

$$\ell' = \max\{A_j\ (j > k), A_k-1, A_j+2\ (j < k)\}.$$

We have $A_k \leqslant \ell, \ell'+1$ and hence $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_k+1 \geqslant -(\ell+\ell'-1)/2$. If $\mathrm{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k = \ell = \ell'+1$, and $n_f = k$. Hence we obtain

(c) (2), and $\widetilde{F}_{-k}(\mathfrak{m}) = \mathfrak{m} - \langle -k+2, k\rangle + \langle -k, k\rangle$. Hence $\xi(\widetilde{F}_{-k}(\mathfrak{m})) = m_{-k,k} + 1 > m_{-k,k} = \xi(\mathfrak{m}')$. Hence we obtain (c) (1).

(ii) If $m_{-k+2,k}$ is even, then $F^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k}(1+q\mathbf{A}_0)$ and

$$\ell' = \max\{A_j \ (j > k), A_k + 1, A_j + 2 \ (j < k)\}.$$

If $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, then $\ell = A_k$ and (a) is satisfied. Assume $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$. We have $A_k \leqslant \ell, \ell' - 1$ and hence $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_k \geqslant -(\ell+\ell'-1)/2$. If $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k = \ell = \ell' - 1$, and hence $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, which is a contradiction.

Case 4. $\mathfrak{m}' = \mathfrak{m} - \delta_{i \neq k}\langle i, k-2\rangle + \langle i, k\rangle$ for $-k+2 < i \leqslant k$. We have

$$\begin{aligned} F^{-k}_{\mathfrak{m},\mathfrak{m}'} &= [m_{i,k}+1] q^{\sum_{j>k}(m_{-k+2,j} - m_{-k,j}) + 2m_{-k+2,k-2} - 2m_{-k,k} + \sum_{-k+2<j<i}(m_{j,k-2} - m_{j,k})} \\ &\in q^{-A_{i-2}}(1+q\mathbf{A}_0), \end{aligned}$$

and

$$\ell' = \max\{A_j \ (j \geqslant k), A_j \ (j < i-2), A_{i-2}+1, A_j + 2 \ (i-2 < j \leqslant k-2)\}.$$

If $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, then $\ell = A_{i-2}$ and (a) holds. Assume $\mathfrak{m}' \neq \widetilde{F}_{-k}(\mathfrak{m})$. Since $A_{i-2} \leqslant \ell, \ell'-1$, we have $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_{i-2} \geqslant -(\ell+\ell'-1)/2$. Hence we obtain (b). If $\operatorname{ord}(F^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then we have $A_{i-2} = \ell = \ell' - 1$. Hence $\mathfrak{m}' = \widetilde{F}_{-k}(\mathfrak{m})$, which is a contradiction.

Q.E.D.

Proposition 4.13. *Suppose $k > 0$. The conditions* (4.10), (4.12), *and* (4.15) *hold, namely, we have*

(a) *if $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{1-\ell}(1+q\mathbf{A}_0)$,*

(b) *if $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$, then $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) \geqslant 1 - \ell + e(\ell - 1 - \ell') = -(\ell+\ell'-1)/2$,*

(c) *if $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$, $\ell \leqslant \ell' + 1$ and $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $b \notin B''$.*

Proof. The proof is similar to the one of the above proposition.

We shall write A_j for $A^{-k}_j(\mathfrak{m})$. Let n_e be as in Definition 4.1 (ii).

Note that $E^{-k}_{\mathfrak{m},\widetilde{E}_{-k}(\mathfrak{m})} \neq 0$ if $\widetilde{E}_{-k}(\mathfrak{m}) \neq 0$. If $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \neq 0$, we have the following five cases.

Case 1. $\mathfrak{m}' = \mathfrak{m} - \langle -k, n\rangle + \langle -k+2, n\rangle$ for $n > k$.

In this case, we have

$$E^{-k}_{\mathfrak{m},\mathfrak{m}'} = (1-q^2)[m_{-k+2,n}+1] q^{1+\sum_{j\geqslant n}(m_{-k+2,j} - m_{-k,j})} \in q^{1-A_n}(1+q\mathbf{A}_0)$$

and

$$\begin{aligned} &\ell = \max\{A_j (j \geqslant -k+2)\}, \\ &\ell' = \max\{A_j \ (j > n), A_n - 1, A_j - 2 \ (j < n)\}. \end{aligned}$$

If $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $\ell = A_n$ and we obtain (a). Assume $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$. Since $A_n \leqslant \ell, \ell'+1$, we have $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 1 - A_n \geqslant -(\ell+\ell'-1)/2$. Hence we obtain (b). If $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then we have $A_n = \ell = \ell' + 1$. Since $A_j \leqslant \ell' = A_n - 1$ for $j > n$, we have $n_e = n$ and $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, which is a contradiction.

Case 2. $\mathfrak{m}' = \mathfrak{m} - \langle -k, k\rangle + \langle -k+2, k\rangle$.

In this case we have

$$\begin{aligned} E^{-k}_{\mathfrak{m},\mathfrak{m}'} &= (1-q^2)[m_{-k+2,k}+1] q^{1+\sum_{j>k}(m_{-k+2,j} - m_{-k,j}) + m_{-k+2,k} - 2m_{-k,k}} \\ &\in q^{1-A_k+\delta(\mathfrak{m}_{-k+2,k} \text{ is odd})}(1+q\mathbf{A}_0). \end{aligned}$$

(i) Assume that $m_{-k+2,k}$ is odd. Then $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$, $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{2-A_k}(1+q\mathbf{A}_0)$ and

$$\ell' = \max\{A_j\ (j>k), A_k-3, A_j-2\ (j<k)\}.$$

We have $A_k \leqslant \ell, \ell'+3$ and hence $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 2-A_k \geqslant -(\ell+\ell'-1)/2$. Hence (b) holds. If $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k = \ell = \ell'+3$. Hence $\ell > \ell'+1$ and (c) holds.

(ii) Assume that $m_{-k+2,k}$ is even. Then $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{1-A_k}(1+q\mathbf{A}_0)$ and

$$\ell' = \max\{A_j\ (j>k), A_k-1, A_j-2\ (j<k)\}.$$

If $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $\ell = A_k$, and we obtain (a). Assume $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$. We have $A_k \leqslant \ell, \ell'+1$ and hence $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 1-A_k \geqslant -(\ell+\ell'-1)/2$. If $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k = \ell = \ell'+1$ and $n_e = k$. Hence $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, which is a contradiction.

Case 3. $\mathfrak{m}' = \mathfrak{m} - \langle -k+2, k\rangle + \delta_{k\neq 1}\langle -k+2, k-2\rangle$. If $k \neq 1$, we have

$$\begin{aligned} E^{-k}_{\mathfrak{m},\mathfrak{m}'} &= (1-q^2)[2(m_{-k+2,k-2}+1)]q^{1+\sum_{j>k}(m_{-k+2,j}-m_{-k,j})+2m_{-k+2,k-2}-2m_{-k,k}} \\ &\in q^{-A_k+\delta(m_{-k+2,k}\text{ is odd})}(1+q\mathbf{A}_0). \end{aligned}$$

If $k=1$, we have

$$E^{-k}_{\mathfrak{m},\mathfrak{m}'} = q^{\sum_{j>k}(m_{-k+2,j}-m_{-k,j})-2m_{-k,k}} = q^{-A_k+\delta(m_{-k+2,k}\text{ is odd})}.$$

In the both cases, we have

$$E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k+\delta(m_{-k+2,k}\text{ is odd})}(1+q\mathbf{A}_0).$$

(i) If $m_{-k+2,k}$ is odd, then $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{1-A_k}(1+q\mathbf{A}_0)$ and

$$\ell' = \max\{A_j\ (j>k), A_k-1, A_j-2\ (j<k)\}.$$

If $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $\ell = A_k$ and (a) is satisfied. We have $A_k \leqslant \ell, \ell'+1$ and hence $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 1-A_k \geqslant -(\ell+\ell'-1)/2$. Assume $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$. If $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k = \ell = \ell'+1$, and $n_e = k$. Hence $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, which is a contradiction.

(ii) If $m_{-k+2,k}$ is even, then $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$, $E^{-k}_{\mathfrak{m},\mathfrak{m}'} \in q^{-A_k}(1+q\mathbf{A}_0)$, and

$$\ell' = \max\{A_j\ (j>k), A_k+1, A_j-2\ (j<k)\}.$$

We have $A_k \leqslant \ell, \ell'-1$ and hence $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -A_k \geqslant -(\ell+\ell'-1)/2$. Hence we obtain (b). If $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then $A_k = \ell = \ell'-1$. Hence $n_e(\mathfrak{m}) \geqslant k$ and $m_{-k+2,k}(\mathfrak{m})$ is even. Hence $\mathfrak{m} \notin B''$.

Case 4. $\mathfrak{m}' = \mathfrak{m} - \langle i, k\rangle + \langle i, k-2\rangle$ for $-k+2 < i \leqslant k-2$.

We have

$$\begin{aligned} E^{-k}_{\mathfrak{m},\mathfrak{m}'} &= (1-q^2)[m_{i,k-2}+1]q^{1+\sum_{j>k}(m_{-k+2,j}-m_{-k,j})+2m_{-k+2,k-2}-2m_{-k,k}+\sum_{-k+2<j\leqslant i}(m_{j,k-2}-m_{j,k})} \\ &\in q^{1-A_{i-2}}(1+q\mathbf{A}_0), \end{aligned}$$

and

$$\ell' = \max\{A_j\ (j\geqslant k),\ A_j\ (j<i-2),\ A_{i-2}-1,\ A_j-2\ (i\leqslant j\leqslant k-2)\}.$$

If $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $\ell = A_{i-2}$ and (a) holds. Assume $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$. Since $A_{i-2} \leqslant \ell, \ell'+1$, we have $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 1-A_{i-2} \geqslant -(\ell+\ell'-1)/2$. Hence we obtain (b). If $\mathrm{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell+\ell'-1)/2$, then we have $A_{i-2} = \ell = \ell'+1$. Hence $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, which is a contradiction.

Case 5. $k \neq 1$ and $\mathfrak{m}' = \mathfrak{m} - \langle k \rangle$. In this case,

$$\begin{aligned} E^{-k}_{\mathfrak{m},\mathfrak{m}'} &= q^{\sum_{j>k}(m_{-k+2,j}-m_{-k,j})-2m_{-k,k}+1-m_{k,k}+2m_{-k+2,k-2}+\sum_{-k+2<i\leqslant k-2}(m_{i,k-2}-m_{i,k})} \\ &\in q^{1-A_{k-2}}(1+q\mathbf{A}_0), \end{aligned}$$

and

$$\ell' = \max\{A_j \ (j \neq k-2), A_{k-2} - 1\}.$$

If $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, then $\ell = A_{k-2}$ and (a) holds. Assume $\mathfrak{m}' \neq \widetilde{E}_{-k}(\mathfrak{m})$. Since $A_{k-2} \leqslant \ell, \ell' + 1$, we have $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = 1 - A_{k-2} \geqslant -(\ell + \ell' - 1)/2$. Hence we obtain (b). If $\operatorname{ord}(E^{-k}_{\mathfrak{m},\mathfrak{m}'}) = -(\ell + \ell' - 1)/2$, then we have $A_{k-2} = \ell = \ell' + 1$. Hence $\mathfrak{m}' = \widetilde{E}_{-k}(\mathfrak{m})$, which is a contradiction. Q.E.D.

Proposition 4.14. *Let $k \in I_{>0}$. Then the conditions in* Corollary 4.11 *holds for $\widetilde{E}_k$, $\widetilde{F}_k$ and ε_k, with the same functions c, e, f.*

Since the proof is similar to and simpler than the one of the preceding two propositions, we omit the proof.

As a corollary we have the following result. We write ϕ for the generator ϕ_0 of $V_\theta(0)$ for short.

Theorem 4.15. (i) *The morphism*

$$\widetilde{V}_\theta(0) := U_q^-(\mathfrak{g}) / \sum_{k\in I} U_q^-(\mathfrak{g})(f_k - f_{-k}) \to V_\theta(0)$$

is an isomorphism.

(ii) $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ *is a basis of the* $\mathbf{K}$*-vector space* $V_\theta(0)$.

(iii) *Set*

$$L_\theta(0) := \sum_{\ell\geqslant 0,\, i_1,\dots,i_\ell\in I} \mathbf{A}_0 \widetilde{F}_{i_1}\cdots\widetilde{F}_{i_\ell}\phi \subset V_\theta(0),$$

$$\mathrm{B}_\theta(0) = \left\{ \widetilde{F}_{i_1}\cdots\widetilde{F}_{i_\ell}\phi \bmod qL_\theta(0) \mid \ell \geqslant 0, i_1,\dots,i_\ell \in I \right\}.$$

Then, $\mathrm{B}_\theta(0)$ *is a basis of* $L_\theta(0)/qL_\theta(0)$ *and* $(L_\theta(0), \mathrm{B}_\theta(0))$ *is a crystal basis of* $V_\theta(0)$*, and the crystal structure coincides with the one of* $\mathcal{M}_\theta$.

(iv) *More precisely, we have*

(a) $L_\theta(0) = \bigoplus_{\mathfrak{m}\in\mathcal{M}_\theta} \mathbf{A}_0 P_\theta(\mathfrak{m})\phi$,

(b) $\mathrm{B}_\theta(0) = \{P_\theta(\mathfrak{m})\phi \bmod qL_\theta(0) \mid \mathfrak{m} \in \mathcal{M}_\theta\}$,

(c) *for any* $k \in I$ *and* $\mathfrak{m} \in \mathcal{M}_\theta$*, we have*

(1) $\widetilde{F}_k P_\theta(\mathfrak{m})\phi \equiv P_\theta(\widetilde{F}_k(\mathfrak{m}))\phi \bmod qL_\theta(0)$,

(2) $\widetilde{E}_k P_\theta(\mathfrak{m})\phi \equiv P_\theta(\widetilde{E}_k(\mathfrak{m}))\phi \bmod qL_\theta(0)$*, where we understand* $P_\theta(0) = 0$,

(3) $\widetilde{E}_k^n P_\theta(\mathfrak{m})\phi \in qL_\theta(0)$ *if and only if* $n > \varepsilon_k(\mathfrak{m})$.

Proof. Let us recall that $P_\theta(\mathfrak{m})\phi \in V_\theta(0)$ is the image of $\widetilde{P}_\theta(\mathfrak{m}) \in \widetilde{V}_\theta(0)$. By Theorem 3.21, $\{\widetilde{P}_\theta(\mathfrak{m})\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ generates $\widetilde{V}_\theta(0)$. Let us set $\widetilde{L} = \sum_{\mathfrak{m}\in\mathcal{M}_\theta} \mathbf{A}_0 \widetilde{P}_\theta(\mathfrak{m}) \subset \widetilde{V}_\theta(0)$. Then Theorem 4.8 implies that $\widetilde{F}_k\widetilde{P}_\theta(\mathfrak{m}) \equiv \widetilde{P}_\theta(\widetilde{F}_k(\mathfrak{m})) \bmod q\widetilde{L}$ and $\widetilde{E}_k\widetilde{P}_\theta(\mathfrak{m}) \equiv \widetilde{P}_\theta(\widetilde{E}_k(\mathfrak{m})) \bmod q\widetilde{L}$. Hence the similar results hold for $L_0 := \sum_{\mathfrak{m}\in\mathcal{M}_\theta} \mathbf{A}_0 P_\theta(\mathfrak{m})\phi \subset V_\theta(0)$ and $P_\theta(\mathfrak{m})\phi$.

Let us show that

$$\text{(A)} \quad \{P_\theta(\mathfrak{m})\phi \bmod qL_0\}_{\mathfrak{m}\in\mathcal{M}_\theta} \text{ is linearly independent in } L_0/qL_0,$$

by the induction of the θ-weight (see Remark 2.12). Assume that we have a linear relation $\sum_{\mathfrak{m}\in S} a_{\mathfrak{m}} P_\theta(\mathfrak{m})\phi \equiv 0 \bmod qL_0$ for a finite subset S and $a_{\mathfrak{m}} \in \mathbb{Q}\setminus\{0\}$. We may assume that all $\mathfrak{m}$ in S have the same θ-weight. Take $\mathfrak{m}_0 \in S$. If $\mathfrak{m}_0$ is the empty multisegment $\emptyset$, then $S=\{\emptyset\}$ and $P_\theta(\mathfrak{m}_0)\phi = \phi$ is non-zero, which is a contradiction. Otherwise, there exists k such that $\varepsilon_k(\mathfrak{m}_0) > 0$ by Lemma 4.7. Applying $\widetilde{E}_k$, we have $\sum_{\mathfrak{m}\in S} a_{\mathfrak{m}} \widetilde{E}_k P_\theta(\mathfrak{m})\phi \equiv \sum_{\mathfrak{m}\in S,\, \widetilde{E}_k(\mathfrak{m})\neq 0} a_{\mathfrak{m}} P_\theta(\widetilde{E}_k(\mathfrak{m}))\phi \equiv 0 \bmod qL_0$. Since $\widetilde{E}_k(\mathfrak{m})$ ($\widetilde{E}_k(\mathfrak{m}) \neq 0$) are mutually distinct, we have $a_{\mathfrak{m}_0} = 0$ by the induction hypothesis. It is a contradiction.

Thus we have proved (A). Hence $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ is a basis of $V_\theta(0)$, which implies that $\{\widetilde{P}_\theta(\mathfrak{m})\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ is a basis of $\widetilde{V}_\theta(0)$. Thus we obtain (i) and (ii).

Let us show (iv) (a). Since $\widetilde{F}_{i_1}\cdots\widetilde{F}_{i\ell}\phi \equiv P_\theta(\widetilde{F}_{i_1}\cdots\widetilde{F}_{i\ell}\emptyset)\phi \bmod qL_0$, we have $L_\theta(0)\subset L_0$ and $L_0 \subset L_\theta(0)+qL_0$. Hence Nakayama's lemma implies $L_0 = L_\theta(0)$. The other statements are now obvious. Q.E.D.

5. Global basis of $V_\theta(0)$

5.1. **Integral form of $V_\theta(0)$.** In this section, we shall prove that $V_\theta(0)$ has a lower global basis. In order to see this, we shall first prove that $\{P_\theta(\mathfrak{m})\phi\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ is a basis of the $\mathbf{A}$-module $V_\theta(0)_{\mathbf{A}}$. Recall that $\mathbf{A} = \mathbb{Q}[q,q^{-1}]$, and $V_\theta(0)_{\mathbf{A}} = U_q^-(\mathfrak{gl}_\infty)_{\mathbf{A}}\phi$.

Lemma 5.1. $V_\theta(0)_{\mathbf{A}} = \bigoplus_{\mathfrak{m}\in\mathcal{M}_\theta} \mathbf{A} P_\theta(\mathfrak{m})\phi$.

Proof. It is clear that $\bigoplus_{\mathfrak{m}\in\mathcal{M}_\theta} \mathbf{A}P_\theta(\mathfrak{m})\phi$ is stable by the actions of $F_k^{(n)}$ by Proposition 3.20. Hence we obtain $V_\theta(0)_{\mathbf{A}} \subset \bigoplus_{\mathfrak{m}\in\mathcal{M}_\theta} \mathbf{A}P_\theta(\mathfrak{m})\phi$.

We shall prove $P_\theta(\mathfrak{m})\phi \in U_q^-(\mathfrak{gl}_\infty)_{\mathbf{A}}\phi$. It is well-known that $\langle i,j\rangle^{(m)}$ is contained in $U_q^-(\mathfrak{gl}_\infty)_{\mathbf{A}}$, which is also seen by Proposition 3.20 (3). We divide $\mathfrak{m}$ as $\mathfrak{m} = \mathfrak{m}_1 + \mathfrak{m}_2$, where $\mathfrak{m}_1 = \sum_{-j<i\leqslant j} m_{ij}\langle i,j\rangle$ and $\mathfrak{m}_2 = \sum_{k>0} m_k\langle -k,k\rangle$. Then $P_\theta(\mathfrak{m}) = P(\mathfrak{m}_1)P_\theta(\mathfrak{m}_2)$ and $P(\mathfrak{m}_1) \in U_q^-(\mathfrak{gl}_\infty)_{\mathbf{A}}$. Hence we may assume from the beginning that $\mathfrak{m} = \sum_{0<k\leqslant a} m_k\langle -k,k\rangle$. We shall show that $P_\theta(\mathfrak{m})\phi \in V_\theta(0)_{\mathbf{A}}$ by the induction on a.

Assume $a>1$. Set $\mathfrak{m}' = \sum_{0<k\leqslant a-4} m_k\langle -k,k\rangle$ and $v = P_\theta(\mathfrak{m}')\phi$. Then $\langle -a+2, a-2\rangle^{[m]}v \in V_\theta(0)_{\mathbf{A}}$ for any m by the induction hypothesis.

We shall show that $\langle -a,a\rangle^{[n]}\langle -a+2,a-2\rangle^{[m]}v$ is contained in $V_\theta(0)_{\mathbf{A}}$ by the induction on n. Since $P_\theta(\mathfrak{m}')$ commutes with $\langle a\rangle$, $\langle -a\rangle$, $\langle -a+2,a-2\rangle$, $\langle -a+2,a\rangle$ and $\langle -a,a\rangle$, Proposition 3.20 (2) implies

$$\begin{aligned}&\langle -a\rangle^{(2n)}\langle -a+2,a-2\rangle^{[n+m]}v\\ &= \sum_{i+j+2t=2n,\ j+t=u} q^{2(n+m)i+j(j-1)/2-i(t+u)}\langle a\rangle^{(i)}\langle -a+2,a\rangle^{(j)}\langle -a,a\rangle^{[t]}\langle -a+2,-2\rangle^{[n+m-u]}v,\end{aligned}$$

which is contained in $V_\theta(0)_{\mathbf{A}}$. Since $\langle a\rangle^{(i)}\langle -a+2,a\rangle^{(j)}\langle -a,a\rangle^{[t]}\langle -a+2,a-2\rangle^{[n+m-u]}v$ is contained in $V_\theta(0)_{\mathbf{A}}$ if $(i,j,t,u)\neq(0,0,n,n)$ by the induction hypothesis on n, $\langle -a,a\rangle^{[n]}\langle -a+2,a-2\rangle^{[m]}v$ is contained in $V_\theta(0)_{\mathbf{A}}$.

If $a=1$, we similarly prove $P_\theta(\mathfrak{m})\phi \in V_\theta(0)_{\mathbf{A}}$ using Proposition 3.20 (1) instead of (2). Q.E.D.

5.2. **Conjugate of the PBW basis.** We will prove that the bar involution is upper triangular with respect to the PBW basis $\{P_\theta(\mathfrak{m})\}_{\mathfrak{m}\in\mathcal{M}_\theta}$.

First we shall prove Theorem 3.10 (4).

For $a,b\in\mathcal{M}$ such that $a\leqslant b$, we denote by $\mathcal{M}_{[a,b]}$ (resp. $\mathcal{M}_{\leqslant b}$) the set of $\mathfrak{m}\in\mathcal{M}$ of the form $\mathfrak{m} = \sum_{a\leqslant i\leqslant j\leqslant b} m_{i,j}\langle i,j\rangle$ (resp. $\mathfrak{m} = \sum_{i\leqslant j\leqslant b} m_{i,j}\langle i,j\rangle$). Similarly we define $(\mathcal{M}_\theta)_{\leqslant b}$.

For a multisegment $\mathfrak{m} \in \mathcal{M}_{\leqslant b}$, we divide $\mathfrak{m}$ into $\mathfrak{m} = \mathfrak{m}_b + \mathfrak{m}_{<b}$, where $\mathfrak{m}_b = \sum_{i \leqslant b} m_{i,j}\langle i, b\rangle$ and $\mathfrak{m}_{<b} = \sum_{i \leqslant j < b} m_{i,j}\langle i, j\rangle$.

Lemma 5.2. *For $n \geqslant 0$ and $a, b \in I$ such that $a \leqslant b$, we have*

$$\overline{\langle a, b\rangle^{(n)}} \in \langle a, b\rangle^{(n)} + \sum_{\mathfrak{m} \underset{\mathrm{cry}}{<} n\langle a,b\rangle} \mathbf{K} P(\mathfrak{m}).$$

Proof. We shall first show

$$\overline{\langle a, b\rangle} \in \langle a, b\rangle + \sum_{a+2 \leqslant k \leqslant b} \langle k, b\rangle U_q^-(\mathfrak{g}) \tag{5.1}$$

by the induction on $b - a$. If $a = b$, it is trivial. If $a < b$, we have

$$\begin{aligned}
\overline{\langle a, b\rangle} &= \langle a\rangle \overline{\langle a+2, b\rangle} - q^{-1}\overline{\langle a+2, b\rangle}\langle a\rangle \\
&\in \langle a\rangle \Big(\langle a+2, b\rangle + \sum_{a+2 < k \leqslant b} \langle k, b\rangle U_q^-(\mathfrak{g})\Big) - q^{-1}\Big(\langle a+2, b\rangle + \sum_{a+2<k\leqslant b} \langle k, b\rangle U_q^-(\mathfrak{g})\Big)\langle a\rangle \\
&\subset \langle a, b\rangle + (q - q^{-1})\langle a+2, b\rangle\langle a\rangle + \sum_{a+2<k\leqslant b} \big(\langle k, b\rangle\langle a\rangle U_q^-(\mathfrak{g}) + \langle k, b\rangle U_q^-(\mathfrak{g})\big).
\end{aligned}$$

Hence we obtain (5.1). We shall show the lemma by the induction on n. We may assume $n > 0$ and

$$\overline{\langle a, b\rangle^{n-1}} \in \langle a, b\rangle^{n-1} + \sum_{\mathfrak{m} \underset{\mathrm{cry}}{<} (n-1)\langle a,b\rangle} \mathbf{K} P(\mathfrak{m}).$$

Hence we have

$$\overline{\langle a, b\rangle^{n}} = \overline{\langle a, b\rangle}\ \overline{\langle a, b\rangle^{n-1}} \in \langle a, b\rangle^{n} + \sum_{a<k\leqslant b} \langle k, b\rangle U_q^-(\mathfrak{g}) + \sum_{\mathfrak{m} \underset{\mathrm{cry}}{<} (n-1)\langle a,b\rangle} \mathbf{K}\langle a, b\rangle P(\mathfrak{m}).$$

For $a < k \leqslant b$ and $\mathfrak{m} \in \mathcal{M}$ such that $\mathrm{wt}(\mathfrak{m}) = \mathrm{wt}(n\langle a, b\rangle) - \mathrm{wt}(\langle k, b\rangle)$, we have $\mathfrak{m} \in \mathcal{M}_{[a,b]}$ and $\mathfrak{m}_b = \sum_{a \leqslant i \leqslant b} m_{i,b}\langle i, b\rangle$ with $\sum_i m_{i,b} = n - 1$. In particular, $m_{a,b} \leqslant n - 1$. Hence $\langle k, b\rangle P(\mathfrak{m}) \in \mathbf{K}P(\mathfrak{m} + \langle k, b\rangle)$ and $\mathfrak{m} + \langle k, b\rangle \underset{\mathrm{cry}}{<} n\langle a, b\rangle$.

If $\mathfrak{m} \underset{\mathrm{cry}}{<} (n-1)\langle a, b\rangle$, then $\langle a, b\rangle P(\mathfrak{m}) \in \mathbf{K}P(\langle a, b\rangle + \mathfrak{m})$ and $\langle a, b\rangle + \mathfrak{m} \underset{\mathrm{cry}}{<} n\langle a, b\rangle$. Q.E.D.

Proposition 5.3. *For $\mathfrak{m} \in \mathcal{M}$,*

$$\overline{P(\mathfrak{m})} \in P(\mathfrak{m}) + \sum_{\mathfrak{n} \underset{\mathrm{cry}}{<} \mathfrak{m}} \mathbf{K} P(\mathfrak{n}).$$

Proof. Put $\mathfrak{m} = \sum_{i \leqslant j \leqslant b} m_{i,j}\langle i, j\rangle$ and divide $\mathfrak{m} = \mathfrak{m}_b + \mathfrak{m}_{<b}$. We prove the claim by the induction on b and the number of segments in $\mathfrak{m}_b$. Suppose $\mathfrak{m}_b = m\langle a, b\rangle + \mathfrak{m}_1$ with $m = m_{a,b} > 0$, where $\mathfrak{m}_1 = \sum_{a<i\leqslant b} m_{i,b}\langle i, b\rangle$.
(i) Let us first show that

$$\overline{P(\mathfrak{m}_b)} \in P(\mathfrak{m}_b) + \sum_{\mathfrak{m}' \underset{\mathrm{cry}}{<} \mathfrak{m}_b} \mathbf{K} P(\mathfrak{m}'). \tag{5.2}$$

We have $\overline{P(\mathfrak{m}_b)} = \overline{P(\mathfrak{m}_1)} \cdot \overline{\langle a,b\rangle^{(m)}}$. Since $\overline{P(\mathfrak{m}_1)} \in P(\mathfrak{m}_1) + \sum_{\mathfrak{m}'_1 \underset{\mathrm{cry}}{<} \mathfrak{m}_1} \mathbf{K}P(\mathfrak{m}'_1)$ by the induction hypothesis, and $\overline{\langle a,b\rangle^{(m)}} \in \langle a,b\rangle^{(m)} + \sum_{\mathfrak{m}'' \underset{\mathrm{cry}}{<} m\langle a,b\rangle} \mathbf{K}P(\mathfrak{m}'')$, we have

$$\overline{P(\mathfrak{m}_b)} \in P(\mathfrak{m}_b) + \sum_{\mathfrak{m}'_1 \underset{\mathrm{cry}}{<} \mathfrak{m}_1,\, \mathfrak{m}'_1 \in \mathcal{M}_{[a+2,b]}} \mathbf{K}P(\mathfrak{m}'_1)\langle a,b\rangle^{(m)} + \sum_{\mathfrak{m}'_1 \underset{\mathrm{cry}}{\leqslant} \mathfrak{m}_1,\, \mathfrak{m}'' \underset{\mathrm{cry}}{<} m\langle a,b\rangle} \mathbf{K}P(\mathfrak{m}'_1)P(\mathfrak{m}'').$$

If $\mathfrak{m}'_1 \underset{\mathrm{cry}}{<} \mathfrak{m}_1$ and $\mathfrak{m}'_1 \in \mathcal{M}_{[a+2,b]}$, then $P((\mathfrak{m}'_1)_{<b})$ and $\langle a,b\rangle^{(m)}$ commute. Hence $P(\mathfrak{m}'_1)\langle a,b\rangle^{(m)} = P(\mathfrak{m}'_1 + m\langle a,b\rangle)$ and $\mathfrak{m}'_1 + m\langle a,b\rangle \underset{\mathrm{cry}}{<} \mathfrak{m}_b$.

If $\mathfrak{m}'_1 \underset{\mathrm{cry}}{\leqslant} \mathfrak{m}_1$, $\mathfrak{m}'_1 \in \mathcal{M}_{[a+2,b]}$ and $\mathfrak{m}'' \underset{\mathrm{cry}}{<} m\langle a,b\rangle$, then we can write $\mathfrak{m}''_b = j\langle a,b\rangle + \mathfrak{m}_2$ with $j < m$ and $\mathfrak{m}_2 \in \mathcal{M}_{[a+2,b]}$. Hence we have

$$P(\mathfrak{m}'_1)P(\mathfrak{m}'') \in \mathbf{K}P((\mathfrak{m}'_1)_b)P(j\langle a,b\rangle)P((\mathfrak{m}'_1)_{<b})P(\mathfrak{m}_2)P(\mathfrak{m}''_{<b}).$$

Since $(\mathfrak{m}'_1)_{<b}, \mathfrak{m}_2 \in \mathcal{M}_{[a+2,b]}$ we have $P((\mathfrak{m}'_1)_{<b})P(\mathfrak{m}_2)P(\mathfrak{m}''_{<b}) \in \sum_{\mathfrak{n}_b \in \mathcal{M}_{[a+2,b]}} \mathbf{K}P(\mathfrak{n})$. Hence we have $P(\mathfrak{m}'_1)P(\mathfrak{m}'') \in \sum_{\mathfrak{n}_b \in \mathcal{M}_{[a+2,b]}} \mathbf{K}P((\mathfrak{m}'_1)_b + j\langle a,b\rangle + \mathfrak{n})$ and $(\mathfrak{m}'_1)_b + j\langle a,b\rangle + \mathfrak{n} \underset{\mathrm{cry}}{<} \mathfrak{m}_b$. Hence we obtain (5.2).

(ii) By the induction hypothesis, $\overline{P(\mathfrak{m}_{<b})} \in P(\mathfrak{m}_{<b}) + \sum_{\mathfrak{m}'' \underset{\mathrm{cry}}{<} \mathfrak{m}_{<b}} \mathbf{K}P(\mathfrak{m}'')$. Since $\overline{P(\mathfrak{m})} = \overline{P(\mathfrak{m}_b)}\ \overline{P(\mathfrak{m}_{<b})}$, (5.2) implies that

$$\overline{P(\mathfrak{m})} \in P(\mathfrak{m}) + \sum_{\mathfrak{m}' \underset{\mathrm{cry}}{<} \mathfrak{m}_b, \mathfrak{m}'' \in \mathcal{M}_{<b}} \mathbf{K}P(\mathfrak{m}')P(\mathfrak{m}'') + \sum_{\mathfrak{m}'' \underset{\mathrm{cry}}{<} \mathfrak{m}_{<b}} \mathbf{K}P(\mathfrak{m}_b)P(\mathfrak{m}'').$$

For $\mathfrak{m}' \underset{\mathrm{cry}}{<} \mathfrak{m}_b$ and $\mathfrak{m}'' \in \mathcal{M}_{<b}$, we have

$$P(\mathfrak{m}')P(\mathfrak{m}'') = P(\mathfrak{m}'_b)P(\mathfrak{m}'_{<b})P(\mathfrak{m}'') \in \sum_{\mathfrak{n} \in \mathcal{M}_{\leqslant b},\, \mathfrak{n}_b = \mathfrak{m}'_b} \mathbf{K}P(\mathfrak{n}) \subset \sum_{\mathfrak{n} \underset{\mathrm{cry}}{<} \mathfrak{m}} \mathbf{K}P(\mathfrak{n}).$$

For $\mathfrak{m}'' \underset{\mathrm{cry}}{<} \mathfrak{m}_{<b}$, we have $P(\mathfrak{m}_b)P(\mathfrak{m}'') = P(\mathfrak{m}_b + \mathfrak{m}'')$ and $\mathfrak{m}_b + \mathfrak{m}'' \underset{\mathrm{cry}}{<} \mathfrak{m}$. Thus we obtain the desired result. Q.E.D.

Proposition 5.4. *For $\mathfrak{m} \in \mathcal{M}_\theta$, we have*

$$\overline{P_\theta(\mathfrak{m})}\phi \in P_\theta(\mathfrak{m})\phi + \sum_{\mathfrak{m}' \in \mathcal{M}_\theta, \mathfrak{m}' \underset{\mathrm{cry}}{<} \mathfrak{m}} \mathbf{K}P_\theta(\mathfrak{m}')\phi.$$

Proof. First note that

$$P(\mathfrak{m})\phi \in \sum_{\mathfrak{n} \in (\mathcal{M}_\theta)_{\leqslant b}} \mathbf{K}P_\theta(\mathfrak{n})\phi \quad \text{for any } b \in I_{>0} \text{ and } \mathfrak{m} \in \mathcal{M}_{[-b,b]}, \tag{5.3}$$

by the weight consideration.

For $\mathfrak{m} \in \mathcal{M}_\theta$, $P_\theta(\mathfrak{m})$ and $P(\mathfrak{m})$ are equal up to a multiple of bar-invariant scalar. Thus we have

$$\overline{P_\theta(\mathfrak{m})} \in P_\theta(\mathfrak{m}) + \sum_{\mathfrak{m}' \in \mathcal{M},\, \mathfrak{m}' \underset{\mathrm{cry}}{<} \mathfrak{m}} \mathbf{K}P(\mathfrak{m}')$$

by Proposition 5.3. Hence it is enough to show that

$$P(\mathfrak{m}')\phi \in \sum_{\mathfrak{n} \in \mathcal{M}_\theta,\, \mathfrak{n} \underset{\mathrm{cry}}{<} \mathfrak{m}} \mathbf{K}P_\theta(\mathfrak{n})\phi \tag{5.4}$$

for $\mathfrak{m}' \in \mathcal{M}$ such that $\mathfrak{m}' \underset{\text{cry}}{<} \mathfrak{m}$ and $\mathrm{wt}(\mathfrak{m}') = \mathrm{wt}(\mathfrak{m})$. Put $\mathfrak{m} = \sum_{i \leqslant j \leqslant b} m_{i,j}\langle i,j\rangle$ and write $\mathfrak{m} = \mathfrak{m}_b + \mathfrak{m}_{<b}$. We prove (5.4) by the induction on b. By the assumption on $\mathfrak{m}'$, we have $\mathfrak{m}' \in \mathcal{M}_{[-b,b]}$ and $\mathfrak{m}'_b \underset{\text{cry}}{\leqslant} \mathfrak{m}_b$. Thus $\mathfrak{m}'_b \in \mathcal{M}_\theta$. Hence $\mathbf{K}P(\mathfrak{m}')\phi = \mathbf{K}P_\theta(\mathfrak{m}'_b)P(\mathfrak{m}'_{<b})\phi$.

If $\mathfrak{m}'_b = \mathfrak{m}_b$, then $\mathfrak{m}'_{<b} <_{\text{cry}} \mathfrak{m}_{<b}$, and the induction hypothesis implies $P(\mathfrak{m}'_{<b})\phi \in \sum_{\mathfrak{n}\in\mathcal{M}_\theta,\ \mathfrak{n} \underset{\text{cry}}{<} \mathfrak{m}_{<b}} \mathbf{K}P_\theta(\mathfrak{n})\phi$. Since $P_\theta(\mathfrak{m}'_b)P_\theta(\mathfrak{n}) = P_\theta(\mathfrak{m}'_b + \mathfrak{n})$ and $\mathfrak{m}'_b + \mathfrak{n} \underset{\text{cry}}{<} \mathfrak{m}$, we obtain (5.4).

If $\mathfrak{m}'_b \underset{\text{cry}}{<} \mathfrak{m}_b$, write $\mathfrak{m}' = \sum_{-b\leqslant i\leqslant j\leqslant b} m'_{i,j}\langle i,j\rangle$. Set $s = m_{-b.b} - m'_{-b,b} \geqslant 0$. Since $\mathrm{wt}(\mathfrak{m}') = \mathrm{wt}(\mathfrak{m})$, we have $\sum_{j<b} m'_{-b,j} = s$. If $s = 0$, then $\mathfrak{m}'_{<b} \in \mathcal{M}_{[-b+2,b-2]}$, and $P(\mathfrak{m}'_{<b})\phi \in \sum_{\mathfrak{n}\in(\mathcal{M}_\theta)_{<b}} \mathbf{K}P_\theta(\mathfrak{n})\phi$ by (5.3). Then (5.4) follows from $\mathfrak{m}'_b + \mathfrak{n} \underset{\text{cry}}{<} \mathfrak{m}$.

Assume $s > 0$. Since $\mathfrak{m}'_{<b} \in \mathcal{M}_{[-b,b]}$, we have $P(\mathfrak{m}'_{<b})\phi \in \sum_{\mathfrak{n}\in(\mathcal{M}_\theta)_{\leqslant b}} \mathbf{K}P_\theta(\mathfrak{n})\phi$ by (5.3). We may assume $(1+\theta)\,\mathrm{wt}(\mathfrak{m}'_{<b}) = (1+\theta)\,\mathrm{wt}(\mathfrak{n})$ (see Remark 2.12). Hence, we have $s = 2m_{-b,b}(\mathfrak{n}) + \sum\limits_{-b<i\leqslant b} m_{i,b}(\mathfrak{n})$. In particular, $m_{-b,b}(\mathfrak{n}) \leqslant s/2$. We have $\mathfrak{m}'_b + \mathfrak{n} \in \mathcal{M}_\theta$ and $P_\theta(\mathfrak{m}'_b)P_\theta(\mathfrak{n})\phi = P_\theta(\mathfrak{m}'_b + \mathfrak{n})\phi$. Since $m_{-b,b}(\mathfrak{m}'_b + \mathfrak{n}) \leqslant (m_{-b,b} - s) + s/2 < m_{-b,b}$, we have $\mathfrak{m}'_b + \mathfrak{n} \underset{\text{cry}}{<} \mathfrak{m}$. Hence we obtain (5.4). Q.E.D.

5.3. **Existence of a global basis.** As a consequence of the preceding subsections, we obtain the following theorem.

Theorem 5.5. (i) $(L_\theta(0), L_\theta(0)^-, V_\theta(0)_{\mathbf{A}})$ *is balanced.*

(ii) *For any* $\mathfrak{m} \in \mathcal{M}_\theta$, *there exists a unique* $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m}) \in L_\theta(0) \cap V_\theta(0)_{\mathbf{A}}$ *such that* $\overline{\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})} = \mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})$ *and* $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m}) \equiv P_\theta(\mathfrak{m})\phi \mod qL_\theta(0)$.

(iii) $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m}) \in P_\theta(\mathfrak{m})\phi + \sum_{\mathfrak{n} \underset{\text{cry}}{<} \mathfrak{m}} q\mathbb{Q}[q]P_\theta(\mathfrak{n})\phi$ *for any* $\mathfrak{m} \in \mathcal{M}_\theta$.

(iv) $\{\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})\}_{\mathfrak{m}\in\mathcal{M}_\theta}$ *is a basis of the* $\mathbf{A}$*-module* $V_\theta(0)_{\mathbf{A}}$, *the* $\mathbf{A}_0$*-module* $L_\theta(0)$ *and the* $\mathbf{K}$*-vector space* $V_\theta(0)$.

Proof. We have already seen that $\overline{P_\theta(\mathfrak{m})\phi} = \sum_{\mathfrak{m}' \underset{\text{cry}}{\leqslant} \mathfrak{m}} c_{\mathfrak{m},\mathfrak{m}'}P_\theta(\mathfrak{m}')\phi$ for $c_{\mathfrak{m},\mathfrak{m}'} \in \mathbf{A}$ with $c_{\mathfrak{m},\mathfrak{m}} = 1$. Let us denote by C the matrix $(c_{\mathfrak{m},\mathfrak{m}'})_{\mathfrak{m},\mathfrak{m}'\in\mathcal{M}_\theta}$. Then $\overline{C}C = \mathrm{id}$ and it is well-known that there is a matrix $A = (a_{\mathfrak{m},\mathfrak{m}'})_{\mathfrak{m},\mathfrak{m}'\in\mathcal{M}_\theta}$ such that $\overline{A}C = A$, $a_{\mathfrak{m},\mathfrak{m}'} = 0$ unless $\mathfrak{m}' \underset{\text{cry}}{\leqslant} \mathfrak{m}$, $a_{\mathfrak{m},\mathfrak{m}} = 1$ and $a_{\mathfrak{m},\mathfrak{m}'} \in q\mathbb{Q}[q]$ for $\mathfrak{m}' \underset{\text{cry}}{<} \mathfrak{m}$. Set $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m}) = \sum_{\mathfrak{m}' \underset{\text{cry}}{\leqslant} \mathfrak{m}} a_{\mathfrak{m},\mathfrak{m}'}P_\theta(\mathfrak{m}')\phi$. Then we have $\overline{\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})} = \mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})$ and $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m}) \equiv P_\theta(\mathfrak{m})\phi \mod qL_\theta(0)$. Since $\mathrm{G}^{\mathrm{low}}_\theta(\mathfrak{m})$ is a basis of $V_\theta(0)_{\mathbf{A}}$, we obtain the desired results. Q.E.D.

Errata to "Symmetric crystals and affine Hecke algebras of type B, Proc. Japan Acad., 82, no. 8, 2006, 131–136" :

(i) In Conjecture 3.8, $\lambda = \Lambda_{p_0} + \Lambda_{p_0^{-1}}$ should be read as $\lambda = \sum\limits_{a\in A} \Lambda_a$, where $A = I \cap \{p_0, p_0^{-1}, -p_0, -p_0^{-1}\}$. We thank S. Ariki who informed us that the original conjecture is false.

(ii) In the two diagrams of $B_\theta(\lambda)$ at the end of § 2, λ should be 0.

(iii) Throughout the paper, $A^{(1)}_\ell$ should be read as $A^{(1)}_{\ell-1}$.

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